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MATHEMATICAL CRYSTALLOGRAPHY

AND

THE THEORY OF GROUPS OF MOVEMENTS

BY

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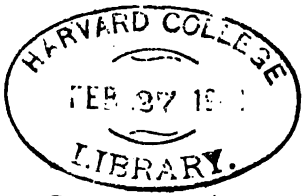

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PREFACE

THE object of this book is to collect for the use of English readers those results of the mathematical theory of crystallography which are not proved in the modern textbooks on that subject in the English language. No claim to originality is made; the theorems proved are in most cases to be found in the works of Schoenflies, Liebisch, &c. From the scope of this treatise are excluded all practical applications of the results obtained (since these are explained at length in the existing English textbooks), and any attempt at a full description of the physical properties of crystals and of the phenomena connected with their growth. On the other hand, I have included a fairly full account of the geometrical theory of crystal-structure which the labours of Bravais, Jordan, Sohncke, Fedorow, Schoenflies, and Barlow have now completed. The last three authors have worked on somewhat different lines; I have in Part II of this book followed the methods and notation of Schoenflies very closely, since they lay the greatest stress on that isomorphism between point-group and space-group which is of such importance in the physical application of the purely geometrical theory. I cannot do better than refer the reader to Schoenflies' "Krystallsysteme und Krystallstructur" where he will find at length what I have presented in a very

similar but shortened form. I have, however, adopted some features from the work of other authors; for instance, the method of chapter xx, §§ 15 and 16, is due to Barlow, the proof of chapter xvi, § 12, is an extension of one due to Jordan, &c. The figures illustrating the groups are on the lines suggested by Fedorow; they have, however, been drawn independently without reference to the diagrams in "Zeitschr. f. Kryst. u. Min." xxiv, which contain some errors. The figures in chapter xix, § 11, are due to Schoenflies, and figures 71, 72, 76, 90, 91, 181, 182, and 183 are also adapted from his "Krystallsysteme und Krystallstructur."

I have not attempted to give a complete list of references, but I have occasionally inserted them when lack of space prevented me from making more than a mere allusion to the contents of the paper quoted.

I take this opportunity of expressing my sincerest gratitude to Professor H. A. Miers, F.R.S., D.Sc., for his generous and sympathetic help. Without his assistance both in preparing the MS. and in correcting the proofs, the book could never have been written; and any merits it may have are due to his suggestions alone.

H. H.

April 16, 1903.

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MATHEMATICAL CRYSTALLOGRAPHY

PART I

CHAPTER I

THE STEREOGRAPHIC PROJECTION.

§ 1. If we take any point O, and join it by straight lines to all the points of a figure U; then the plane figure formed by the intersections of these straight lines with any plane p is called the projection of U on p with respect to O.

The projection of greatest interest in crystallography is the 'stereographic projection.' Before we consider this, however, we shall first prove some properties of 'inversion.'

ERRATA.

Page 16. *Delete the first four lines.*
Page 21. *Add* '&c.' *at the end of the note.*
Page 65, line 8 from bottom. *Add a comma after* s_1''
Page 110. *Add at the end of footnote* §, 'See also a paper by Professor Miers, "Proc. Roy. Soc." lxxi, No. 474 (1903), p. 441.'

Fig. 1.

close to P, the angle OQP = the angle QPP'; $\therefore$ the angles QPP', $Q'P'P$ are equal when Q is very close to P. If PQ is the tangent at P to a curve, $P'Q'$ will be the tangent at P' to the inverse curve, hence we have

If P, P' are two corresponding points on inverse curves, the tangents at P and P' lie in the same plane and make equal angles with OPP'.

Any two curves cut at the same angle as the two inverse curves.

For let PA, PB be the tangents to two curves at their point of intersection P; and let $P'A$, $P'B$ be the tangents to the

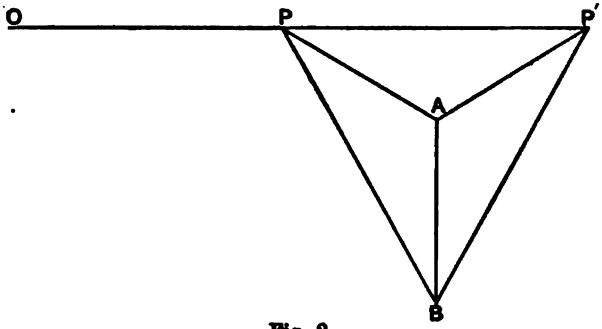

Fig. 2.

two inverse curves at the corresponding point P' (where the planes PAP', PBP' may or may not coincide).

Then $\because$ the angle APP' = the angle $AP'P$, $\therefore$ $AP = AP'$;

And $\because$ the angle BPP' = the angle $BP'P$, $\therefore$ $BP = BP'$.

And AB is common to the triangles APB, $AP'B$; $\therefore$ the angle APB = the angle $AP'B$.

The inverse of a sphere is a sphere or a plane.

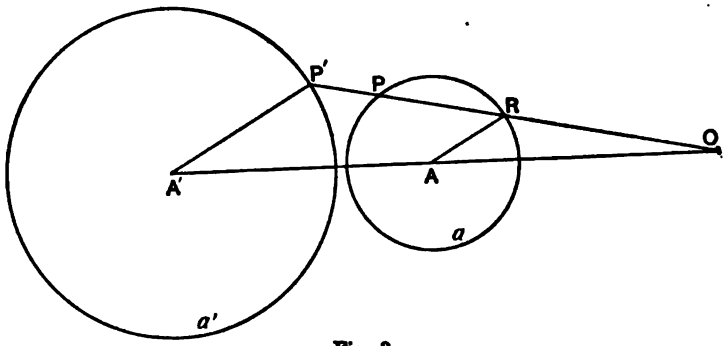

Fig. 3.

Let a be the sphere, and A its centre.

Let the length of the tangent from O to a be h. On OA take a point OA' such that $OA' : OA :: k^2 : h^2$ (Fig. 3).

Let P and P' be inverse points, P being any point on a, and let OP meet a again in R.

Then since $OP.OP' = k^2$, $OP.OR = h^2$

$$\therefore OR : OP' :: h^2 : k^2$$
$$:: OA : OA'.$$

Therefore the triangles OAR, $OA'P'$ are similar; and $\therefore$ $AR : A'P' :: OA : OA'$.

But OA, OA', AR are fixed in length, and $\therefore$ so is $A'P'$; hence, since A' is a fixed point, P' lies on a sphere whose centre is A'.

If O lies on a then A' is infinitely far from A, and the inverse of a is a sphere whose centre is infinitely distant from its surface; i.e. the inverse of a is a plane.

Conversely, *if a', a are any two spheres, a point can be found with respect to which a', a are inverse to one another.*

For (Fig. 3) let A', A be the centres of a', a, and let $A'P'$, AR be any two parallel radii, and let $P'R$ meet $A'A$ in O, and a again in P.

Then $OA : OA' :: AR : A'P'$, and $\therefore$ O is a fixed point. Also since $OR.OP$ is constant, and the ratio $OR : OP' = OA : OA'$ which is constant; $\therefore$ $OP.OP'$ is constant.

Again, *if p is a plane, and a a sphere; a and p are inverse with respect to either point in which a perpendicular to p through the centre of a meets a.*

For (Fig. 4), let A be the centre of a, and draw $OAC'C$ perpendicular to p, meeting a at C' and O, and p at C. Take any point P on p, and join $OP'P$ cutting a at P' and O.

Then $\because$ the angles $C'CP$, $C'P'P$ are right angles; so that P, C, C', P' are concyclic; $\therefore$ $OP'.OP = OC'.OC$ which is constant.

Similarly we may prove that a and p are inverse with respect to C'.

Since the inverse of the intersection of two surfaces is evidently the intersection of the two inverse surfaces, and since the intersection of two spheres is a circle, we see at once that

The inverse of a circle is a circle.

§ 3. *The Stereographic projection.* In this method of projection figures lying on the surface of a sphere are projected from a point O also lying on the surface of the sphere, on to any plane p parallel to the tangent plane at O (Fig. 4).

The projected figure is the inverse of the original with respect to O (§ 2); hence a circle projects into a circle (or

into a straight line if its plane passes through O), and angles are unaltered in magnitude by the projection.

Let s' be the great circle whose plane is parallel to p, and let s be its projection. Then points on the sphere on the side of s' remote from O project into points lying inside s, while points on the same side as O project into points outside s (the projection of points very close to O being at an infinite distance from C, the centre of s); it will however be convenient to represent the latter as follows:—

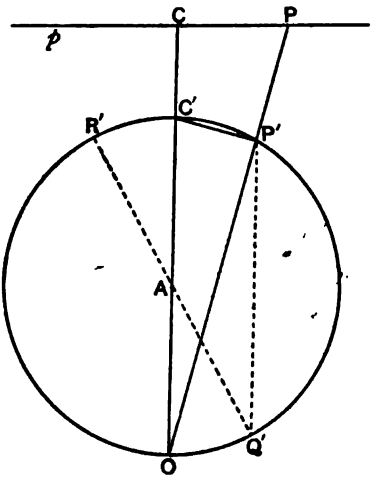

Fig. 4.

Take any point Q' on the same side of s' as O. Draw a line through Q' perpendicular to p to meet the sphere again in P'. Let P be the projection of P'. Then P is taken as representing either P' or Q', and is marked with a cross if it represents P', and with a small circle if it represents Q' (Fig. 5). In this way every point on the sphere is represented by a point lying inside s. If in Fig. 5 P, C, R are collinear and $CP = RC$, it is evident:

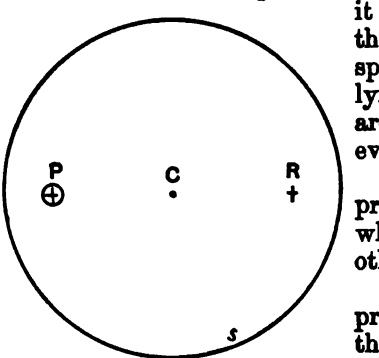

Fig. 5.

(1) that $P(+)$ and $P(\circ)$ represent points on the sphere which are the reflexions of each other in the plane of s'.

(2) That $P(\circ)$ and $R(+)$ represent points which are such that the line joining them is bisected by the centre of the sphere *.

(3) That $P(+)$ and $R(+)$ represent points which are such that the line joining them is bisected at right angles by the line OAC.

§ 4. If on any sphere E, F are the poles of the great circles

* i. e. which are extremities of a diameter of the sphere.

e, f, then any two planes through E and F cut off equal arcs from e and f (Fig. 6).

For let the planes cut e in U, U' and f in V, V' respectively; and let the great circle through E and F cut e and f in X and Y. Then the arc $EU = \frac{\pi}{2}$ = the arc FV; $\therefore$ the straight line EU = the straight line FV.

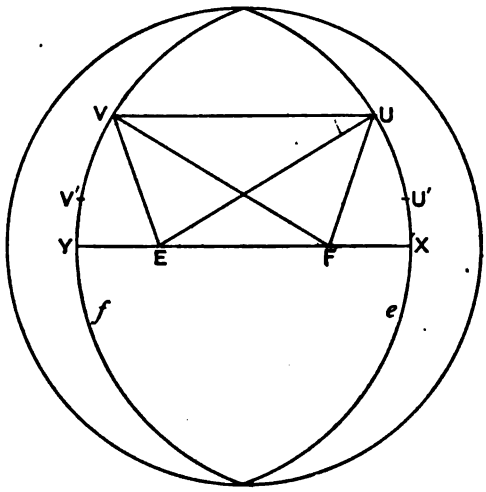

Fig. 6.

Now in the plane triangles EUF, FVE, $EU = FV$, EF is common, and the angle EUF = the angle FVE (for E, V, U, F lie on a small circle); $\therefore$ the angle UEF = the angle VFE* and $FU = EV$, and $\therefore$ the arc FU = the arc EV. Again, since in the spherical triangles EUF, FVE, $EU = FV$, $FU = EV$, and EF is common; therefore the (spherical) angle FEU = the angle EFV.

Moreover, since in the spherical triangles UEX, VFY, $EU = \frac{\pi}{2} = FV$, $EX = \frac{\pi}{2} = FY$, and the angle UEX = the angle VFY; $\therefore$ the arcs UX, VY are equal.

Similarly the arcs $U'X$, $V'Y$ are equal.

Hence the arcs UU', VV' are equal.

Now the projection of any diameter of s' on the plane p with respect to the point O is a diameter of the circle s. But any great circle meets s' at the ends of a diameter; therefore the projection of any great circle meets s at the ends of a diameter.

Consider any two points P, Q on a diameter of s. Let D, S be the ends of the diameter perpendicular to PQ.

* Since the plane $EFUV$ cuts each of the circles e and f in two points, the pair of points U and V may be taken in four different ways. We have taken them in the figure so that EV and FU are not parallel, and therefore the angles UFE, VEF are not supplementary. Then it follows from the reasoning in the text that they are equal; and hence that the angles UEF, VFE are equal.

Join DP, DQ and produce them to meet s again in P_1, Q_1 (Fig. 7).

Let P', Q', D', S', P_1', Q_1' be the points of which P, Q, D, S, P_1, Q_1 are the projections. Then D' is the pole of the great circle through P' and Q', and O is the pole of the great circle $D'P_1'S'$ (i.e. the circle s').

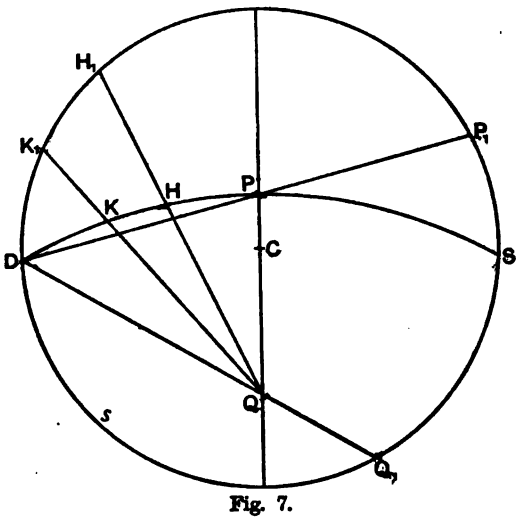

Fig. 7.

Hence, by the above, the planes $D'P'O$, $D'Q'O$ (which are the same as DPO, DQO) cut off equal arcs $P'Q'$, $P_1'Q_1'$; i.e. PQ, P_1Q_1 represent equal arcs.

Again, evidently P_1Q_1 subtends the same angle at C as $P_1'Q_1'$ does at the centre of the sphere.

Hence PQ is the projection of an arc subtending an angle P_1CQ_1 at the centre of the sphere.

In particular, if P_1CQ_1 is a right angle, Q represents the pole of the circle whose projection is the arc DPS.

Again, draw any two straight lines through Q meeting DPS, DP_1S at K, H and K_1, H_1 respectively.

If D', P', S', Q', K', H', K_1', H_1' are the points of which D, P, S, Q, K, H, K_1, H_1 are the projections; and if P_1CQ_1 is a right angle, then Q' is the pole of $D'P'S'$. Also O is the pole of $D'P_1'S'$; and hence the planes $Q'OK'$, $Q'OH'$ (which are the same as QOK, QOH) cut off equal arcs $H'K'$, $H_1'K_1'$; i.e. HK and H_1K_1 represent equal arcs. As before, the angle K_1CH_1 = the angle subtended by $H_1'K_1'$ at the centre of the sphere; and therefore HK is the projection of an arc subtending an angle H_1CK_1 at the centre of the sphere.

§ 5. Hence we have now shown how to find the projection of that arc of any great circle which subtends any given angle at the centre of the sphere.

Suppose for instance we are given the circle DPS of the projection, and a point H on it (Fig. 7), and we wish to find

a point K on the circle such that the arc HK is the projection of an arc subtending an angle α at the centre of the sphere; then we proceed as follows:—

Bisect DS at P, produce DP to meet s again in P_1. Take the point Q_1 on s, such that P_1CQ_1 is a right angle; join DQ_1 cutting PC at Q.

Then Q represents the pole of the circle whose projection is DPS.

Produce QH to meet s at H_1; take K_1 on s such that the angle $H_1CK_1 = \alpha$. Join QK_1 cutting DPS at K. Then K is the point required.

CHAPTER II

PROPERTIES COMMON TO SYMMETRICAL AND ASYMMETRICAL CRYSTALS.

§ 1. *A portion of solid matter is said to be homogeneous if its physical and chemical properties are the same about every point of its substance* *.

Solid homogeneous matter is either amorphous or crystalline.

Solid homogeneous matter is amorphous if its physical properties are the same in every direction; if this is not so the matter is said to be crystalline.

Thus if rods of given length and cross section are cut in various directions from the matter; and if *all* the rods conduct heat and electricity with equal ease, offer an equal resistance to compression, &c., the matter is amorphous; if otherwise it is crystalline: but in either case rods cut in the same direction from different parts of the matter have identical properties, because the matter is homogeneous.

When solid crystalline material is formed in an unconfined space, either from the molten substance, from a vapour, or from a solution, it is found that the material is bounded by plane surfaces forming a polyhedron. This polyhedron is called a *crystal* of the substance; the planes bounding it are called the *faces* of the crystal; and the intersection of any two such planes is called an *edge* of the crystal.

Though the appearance of the crystal may vary very much with the conditions of growth, it is found that the angles between its faces are constant for a given substance, and are characteristic of that substance.

If from a fixed point we draw perpendiculars to all the faces of any crystal of a given substance, we obtain a pencil of lines which is characteristic of that substance.

§ 2. Let planes through a point O within the crystal parallel

* With the exception of points whose distance from the boundary of the substance is comparable with the size of the molecules of which the matter is formed.

to any three faces of the crystal meet in the straight lines OA, OB, OC; and let any other face (called the *parametral* face) meet these lines in A, B, C. Suppose that $OA : OB : OC = a : b : c$.

Let any other face of the crystal cut OA, OB, OC in A', B', C'; and let $OA' : OB' : OC' = \frac{a}{h} : \frac{b}{k} : \frac{c}{l}$.

Then h, k, l are said to be the *indices* of the face $A'B'C'$; the face is represented by the symbol (hkl) and is known as 'the face (hkl).'

This is Miller's notation, and is now universally adopted. We take h positive if A' is on the same side of O as A, otherwise negative; and similarly for k and l.

If $h = 0$ the face $A'B'C'$ is parallel to OA, and similarly for k and l.

At present we shall only concern ourselves with the *ratios* $h : k : l$ so long as h, k, and l do not alter their signs; thus $(h\,k\,l)$, $(xh\,xk\,xl)$ represent the same face if x is positive. A distinction is made, however, between two parallel faces on opposite sides of O; thus $(h\,k\,l)$, $(-h\,-k\,-l)$ represent different (but parallel) faces.

It is usual to take for h, k, l quantities as simple as possible; thus we talk of 'the face (212),' not of 'the face (636)' or the 'face $(\sqrt{3}\ \frac{\sqrt{3}}{2}\ \sqrt{3})$.' We shall see later that it is always possible to express h, k, l as small integers.

In the symbols which represent faces it is customary to print minus signs *above* instead of *before* the letter or number; thus for $(-h\,-k\,-l)$ we write $(\bar{h}\,\bar{k}\,\bar{l})$.

A face parallel to OBC is represented by (100) or $(\bar{1}00)$, so for faces parallel to OCA and OAB.

Again, we are only concerned with the *ratios* $a : b : c$. It is usual to take $b = 1$. These ratios are called *the axial ratios*. The lines OA, OB, OC are called the *crystallographic axes*; they are parallel to 3 crystal edges.

§ 3. *Given the indices of a face referred to one set of crystallographic axes through O, to find the indices referred to any other set of crystallographic axes through O.*

Let $(x'\ y'\ z')$ be the Cartesian coordinates of any point referred to the first set of crystallographic axes, and $(X\ Y\ Z)$ the Cartesian coordinates of the same point referred to the new crystallographic axes.

Let $x' = ax$, $y' = by$, $z' = cz$.

Now X, Y, Z are connected with x', y', z' and therefore with x, y, z by linear relations*;

Let these relations be

$$\left.\begin{aligned} x &= l_1X + m_1Y + n_1Z \\ y &= l_2X + m_2Y + n_2Z \\ z &= l_3X + m_3Y + n_3Z \end{aligned}\right\}.$$

Let

$$\left.\begin{aligned} d_1x + e_1y + f_1z &= 0 \\ d_2x + e_2y + f_2z &= 0 \\ d_3x + e_3y + f_3z &= 0 \end{aligned}\right\}$$

be the equations referred to the old axes of the three planes whose intersections form the new axes; and let $D_1, E_1 \ldots$ be the co-factors of $d_1, e_1 \ldots$ in the determinant

$$\begin{vmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{vmatrix}.$$

Then the intersection of

$$\left.\begin{aligned} d_2x + e_2y + f_2z &= 0 \\ \text{and} \quad d_3x + e_3y + f_3z &= 0 \end{aligned}\right\}; \text{ i.e. } \frac{x}{D_1} = \frac{y}{E_1} = \frac{z}{F_1},$$

or $$\frac{l_1X + m_1Y + n_1Z}{D_1} = \frac{l_2X + m_2Y + n_2Z}{E_1} = \frac{l_3X + m_3Y + n_3Z}{F_1}$$

must be identical with $Y = Z = 0$.

Therefore $\frac{l_1}{D_1} = \frac{l_2}{E_1} = \frac{l_3}{F_1} = \frac{1}{P}$ (say); and similarly

$\frac{m_1}{D_2} = \frac{m_2}{E_2} = \frac{m_3}{F_2} = \frac{1}{Q}$ (say); and $\frac{n_1}{D_3} = \frac{n_2}{E_3} = \frac{n_3}{F_3} = \frac{1}{R}$ (say).

Let (def) be the symbol of any face referred to the old axes; the equation of a parallel plane through O is

$$\frac{x'}{\begin{matrix} a \\ d \end{matrix}} + \frac{y'}{\begin{matrix} b \\ e \end{matrix}} + \frac{z'}{\begin{matrix} c \\ f \end{matrix}} = 0,$$

or $$dx + ey + fz = 0.$$

Its equation referred to the new axes is therefore

$$d(l_1X + m_1Y + n_1Z) + e(l_2X + m_2Y + n_2Z) + f(l_3X + m_3Y + n_3Z) = 0,$$

or $$X(dl_1 + el_2 + fl_3) + Y(dm_1 + em_2 + fm_3) + Z(dn_1 + en_2 + fn_3) = 0,$$

or $$\frac{X}{P}(dD_1 + eE_1 + fF_1) + \frac{Y}{Q}(dD_2 + eE_2 + fF_2) + \frac{Z}{R}(dD_3 + eE_3 + fF_3) = 0.$$

* The nature of these relations is discussed in chapter vii.

Hence if we take $P:Q:R$ as our new axial ratios we must take

$$(dD_1+eE_1+fF_1,\ dD_2+eE_2+fF_2,\ dD_3+eE_3+fF_3)$$

as the new symbol of the face $(d\,e\,f)$.

To obtain $P:Q:R$ we proceed as follows:—

Let α, β, γ be the angles the old axes make with each other. The square of the distance of any point on the line $Y=Z=0$ from O, is X^2, and is also

$$\begin{aligned} &x'^2+y'^2+z'^2+2y'z'\cos\alpha+2z'x'\cos\beta+2x'y'\cos\gamma \\ &= X^2(a^2l_1^2+b^2l_2^2+c^2l_3^2+2bcl_2l_3\cos\alpha+2cal_3l_1\cos\beta \\ &\qquad +2abl_1l_2\cos\gamma) \\ &= \frac{X^2}{P^2}(a^2D_1^2+b^2E_1^2+c^2F_1^2+2bcE_1F_1\cos\alpha+2caF_1D_1\cos\beta \\ &\qquad +2abD_1E_1\cos\gamma) \end{aligned}$$

$$\begin{aligned} \therefore P^2 = (&a^2D_1^2+b^2E_1^2+c^2F_1^2+2bcE_1F_1\cos\alpha \\ &+2caF_1D_1\cos\beta+2abD_1E_1\cos\gamma); \end{aligned}$$

and similar expressions hold for Q^2 and R^2.

§ 4. It is an observed fact that, for *any* face $(d\,e\,f)$ of a crystal, the ratios $d:e:f$ are *rational*; and that in general d, e, f are in the ratios of *small* integers, e. g. the faces liable to occur are $(1\ 1\ 1)$, $(1\ \bar{2}\ 2)$, $(3\ 2\ 1)$, $(\bar{4}\ 2\ \bar{1})$ and so on.

This 'law' is called the *law of rational indices*; and it is on this law combined with the constancy of angles in different crystals of a given substance, that the whole of the argument of the first part of this book is based.

If the law holds for one set of crystallographic axes it holds for any other.

For the ratios $d:e:f$; $d_1:e_1:f_1$; $d_2:e_2:f_2$; $d_3:e_3:f_3$ of the previous section are all rational; hence, since we are only concerned with the *ratios* of d, e, f, &c. (cf. p. 9), we may take $d, e, f, d_1, e_1, f_1, d_2, e_2, f_2, d_3, e_3, f_3$ themselves rational. In this case $D_1, D_2, D_3, E_1, E_2, E_3, F_1, F_2, F_3$ are also rational, and therefore $dD_1+eE_1+fF_1$, $dD_2+eE_2+fF_2$, $dD_3+eE_3+fF_3$, are also rational.

We may sum up the above statements thus:—

If $(h\,k\,l)$ be any face of a crystal, referred to lines through a point parallel to any three edges as crystallographic axes; then $h:k:l=m:n:p$, where m, n, p are integers and in general small integers.

All that is meant is that no measurements which we can make can prove that the indices of any given crystal face are not in the ratios of small integers. The most convenient

and accurate means we possess of finding the indices of the faces of a crystal is that of measuring the angles between the various faces, and all such measurements are liable to a certain range of error. Hence to say merely that 'we cannot prove that the indices of any given face are not in the ratios of integers' would be only equivalent to saying that, 'if we are given any incommensurable quantity r, we can always find a commensurable quantity s such that $r-s$ is as small as we please'; but to say that 'we cannot prove that the indices of any given face are not in the ratios of *small* integers' is to state an experimental law of the highest importance.

We pointed out before that we are only concerned with the *ratios* of the indices of any face; we now see that it is always possible to take the indices as integers. We shall in future take as the indices of any face the *smallest integers possible*, and talk for instance of 'the face $(1\ \bar{2}\ 3)$,' not of 'the face $(2\ \bar{4}\ 6)$,' or 'the face $(\frac{1}{3}\ \bar{\frac{2}{3}}\ 1)$.'

§ 5. If (as in § 3) the Cartesian coordinates of any point referred to the crystallographic axes are $(x'\,y'\,z')$, and if $x'=ax$, $y'=by$, $z'=cz$; then the equations of the planes through the origin O parallel to the faces $(h_1k_1l_1)$ and $(h_2k_2l_2)$ are

$$h_1x+k_1y+l_1z=0$$
$$h_2x+k_2y+l_2z=0.$$

Therefore the equation of the line through O parallel to the edge in which the two faces meet is

$$\frac{x}{k_1l_2-k_2l_1}=\frac{y}{l_1h_2-l_2h_1}=\frac{z}{h_1k_2-h_2k_1}.$$

The quantities $k_1l_2-k_2l_1$, $l_1h_2-l_2h_1$, $h_1k_2-h_2k_1$ are integral since h_1, k_1, l_1, h_2, k_2, l_2 are all integral; the smallest integers proportional to them are called the *indices of the edge* * in which the two faces meet.

If H, K, L are the indices of any edge, it is represented by the symbol $[H\ K\ L]$ and is known as 'the edge $[H\ K\ L]$.'

The direction-ratios of the edge [1 1 1] are evidently proportional to a, b, c.

If $[H_1\ K_1\ L_1]$, $[H_2K_2L_2]$ are any two edges, the equation of the plane through O parallel to both edges is

$$(K_1L_2-K_2L_1)x+(L_1H_2-L_2H_1)y+(H_1K_2-H_2K_1)z=0.$$

This is parallel to the face whose indices are $K_1L_2-K_2L_1$,

* The reader must carefully distinguish between 'indices of a face' and 'indices of an edge.'

$L_1H_2-L_2H_1$, $H_1K_2-H_2K_1$*: that is, to a face whose indices are integral, since H_1, K_1, L_1, H_2, K_2, L_2 are all integral.

We have then,

The intersection of any two possible crystal faces is a possible crystal edge, and a plane parallel to any two possible crystal edges is a possible crystal face,

meaning by a 'possible' crystal face or edge one whose indices are integral.

§ 6. Crystal faces which are all parallel to one edge are said to form a *zone*. A line through O parallel to the edge is called the *zonal axis*; the normals through any point to the faces of a zone evidently lie in a plane perpendicular to the zonal axis.

The equations of planes through O parallel to the faces

$$(h_1k_1l_1),\ (h_2k_2l_2),\ (h_3k_3l_3),\ (h_4k_4l_4)\ldots \text{ are}$$

$$h_1x+k_1y+l_1z=0,\ h_2x+k_2y+l_2z=0,$$
$$h_3x+k_3y+l_3z=0,\ h_4x+k_4y+l_4z=0,\ \&c.;$$

and hence the condition that these faces should form a zone is that all the determinants represented by

$$\left\| \begin{matrix} h_1 & h_2 & h_3 & h_4 & . & . \\ k_1 & k_2 & k_3 & k_4 & . & . \\ l_1 & l_2 & l_3 & l_4 & . & . \end{matrix} \right\|$$

should be equal to zero.

Taking the three sets of indices $h_1k_1l_1$, $h_2k_2l_2$, $h_nk_nl_n$, we have

$$\left| \begin{matrix} h_1 & h_2 & h_n \\ k_1 & k_2 & k_n \\ l_1 & l_2 & l_n \end{matrix} \right| = 0$$

and $\therefore$

$$\left| \begin{matrix} h_1, & h_2, & h_n-\lambda h_1-\mu h_2 \\ k_1, & k_2, & k_n-\lambda k_1-\mu k_2 \\ l_1, & l_2, & l_n-\lambda l_1-\mu l_2 \end{matrix} \right| = 0.$$

Not all the quantities $k_1l_2-k_2l_1$, $l_1h_2-l_2h_1$, $h_1k_2-h_2k_1$, can $= 0$, for that would involve

$$\frac{h_1}{h_2}=\frac{k_1}{k_2}=\frac{l_1}{l_2};$$

suppose the first of them $\neq 0$, and choose λ, μ to satisfy the equations $k_n-\lambda k_1-\mu k_2=l_n-\lambda l_1-\mu l_2=0$.

Then the determinant last written is equal to

$$\left| \begin{matrix} k_1 & k_2 \\ l_1 & l_2 \end{matrix} \right| \times (h_n-\lambda h_1-\mu h_2), \text{ and } \therefore\ h_n-\lambda h_1-\mu h_2=0;$$
$$\text{for } \left| \begin{matrix} k_1 & k_2 \\ l_1 & l_2 \end{matrix} \right| \neq 0.$$

* Or smaller integers proportional to these quantities.

Hence the indices of any face of the zone to which the two faces $(h_1k_1l_1)$ $(h_2k_2l_2)$ belong are of the form

$$(\lambda h_1+\mu h_2,\ \lambda k_1+\mu k_2,\ \lambda l_1+\mu l_2).$$

The converse is also true since

$$\begin{vmatrix} h_1, & h_2, & \lambda h_1+\mu h_2 \\ k_1, & k_2, & \lambda k_1+\mu k_2 \\ l_1, & l_2, & \lambda l_1+\mu l_2 \end{vmatrix} \equiv 0.$$

§ 7. *To find the anharmonic ratio of four faces of a zone.*

Let n_1, n_2, n_3, n_4 be the four faces; and let $[n_i\ n_j]$ denote the angle between n_i and n_j (where i and j are any two of the integers 1, 2, 3, 4)*. Then

$$\sin [n_1\ n_3] \cdot \sin [n_2\ n_4] + \sin [n_1\ n_4] \cdot \sin [n_2\ n_3]$$

is called the anharmonic or cross ratio of the four faces.

Let perpendiculars VN_1, VN_2, VN_3, VN_4 be drawn from any point V to the faces; then evidently the anharmonic ratio of the faces $=$ sin $N_3VN_1 \cdot$ sin $N_4VN_2 +$ sin $N_4VN_1 \cdot$ sin N_3VN_2, the anharmonic ratio of the pencil $V\ (N_1\ N_2\ N_3\ N_4)$ †. We denote this pencil by $[P]$; its anharmonic ratio is evidently independent of the position of N.

Draw four planes through V perpendicular to the four lines of the pencil $[P]$; these are parallel to the crystal faces and meet in a line (the line through V parallel to the zonal axis).

These four planes are cut by the plane of the pencil $[P]$ in a pencil having the same angles, and therefore the same anharmonic ratio as the pencil $[P]$. Now it is a well-known theorem that the pencil in which four planes through a line are cut by *any* plane has a constant anharmonic ratio; hence the anharmonic ratio of the pencil $[P]$ is equal to the anharmonic ratio of the pencil in which four planes through V parallel to the crystal faces are cut by any plane Π.

As before, let OA, OB, OC be the crystallographic axes, and $a:b:c$ the axial ratios. We noticed above that the anharmonic ratio of the pencil $[P]$ is independent of the position of the vertex V, we may therefore take V on the axis OB, and take the plane OBC as the plane Π.

Let $h_1k_1l_1$, $h_2k_2l_2$, $h_3k_3l_3$, $h_4k_4l_4$ be the indices of the four faces; their intercepts on the axes are proportional to

$$\frac{a}{h_1}\ \frac{b}{k_1}\ \frac{c}{l_1},\ \text{\&c.}$$

* This angle is the external angle between the two faces, so that $[n_i\ n_j]=0$ if n_i and n_j coincide.

† This is the definition of anharmonic ratio adopted by crystallographers; it is not quite the same as that commonly used by mathematicians.

Let OC cut the four planes through V parallel to the crystal faces in X_1, X_2, X_3, X_4. Then the pencil $[P]$ has the same anharmonic ratio as the pencil $V(X_1X_2X_3X_4)$; i. e. as the range $(X_1X_2X_3X_4)$. Therefore the anharmonic ratio of the pencil $[P]$

$$= \frac{(OX_3 - OX_1)(OX_4 - OX_2)}{(OX_4 - OX_1)(OX_3 - OX_2)}.$$

Now

$$\frac{OX_1}{OV} = \frac{\frac{c}{l_1}}{\frac{b}{k_1}} = \frac{ck_1}{bl_1}, \text{ \&c.;}$$

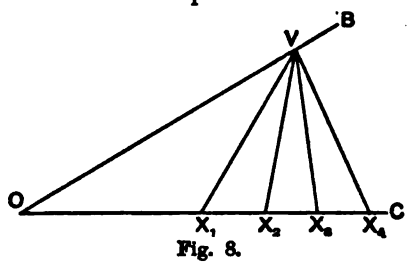

Fig. 8.

hence $$OX_1 = OV \cdot \frac{ck_1}{bl_1}; \; OX_2 = OV \cdot \frac{ck_2}{bl_2};$$

$$OX_3 = OV \cdot \frac{ck_3}{bl_3}; \; OX_4 = OV \cdot \frac{ck_4}{bl_4}.$$

Therefore the anharmonic ratio of the four faces = the anharmonic ratio of the pencil $[P]$,

$$= \frac{\left(\frac{ck_3}{bl_3} - \frac{ck_1}{bl_1}\right)\left(\frac{ck_4}{bl_4} - \frac{ck_2}{bl_2}\right)}{\left(\frac{ck_4}{bl_4} - \frac{ck_1}{bl_1}\right)\left(\frac{ck_3}{bl_3} - \frac{ck_2}{bl_2}\right)} = \frac{(k_1l_3 - k_3l_1)(k_2l_4 - k_4l_2)}{(k_1l_4 - k_4l_1)(k_2l_3 - k_3l_2)}$$

$$= \text{(similarly)} \frac{(l_1h_3 - l_3h_1)(l_2h_4 - l_4h_2)}{(l_1h_4 - l_4h_1)(l_2h_3 - l_3h_2)}$$

$$= \frac{(h_1k_3 - h_3k_1)(h_2k_4 - h_4k_2)}{(h_1k_4 - h_4k_1)(h_2k_3 - h_3k_2)}*.$$

Since h_1, k_1, l_1, &c., are all integral, therefore

the anharmonic ratio of four faces of a zone is rational †.

* For other proofs see Maskelyne's "Morphology of Crystals," pp. 70 and 71; Lewis's "Crystallography," p. 81 (G. Césaro, Rivista di Min. e Crist. Ital., 1889); Liebisch's "Grundriss der physikalischen Krystallographie," pp. 81 and 82, &c. The above proof was given by Hilary Bauerman ("Systematic Mineralogy," 1881, p. 35), and independently by the author ("Min. Mag." XIII, 1900, p. 69).

† This theorem is due to C. Fr. Gauss, 1831.

If the indices of the edge in which n_i and n_j meet are H_{ij}, K_{ij}, L_{ij} we see that

$$\frac{H_{13} \cdot H_{24}}{H_{14} \cdot H_{23}} = \frac{K_{13} \cdot K_{24}}{K_{14} \cdot K_{23}} = \frac{L_{13} \cdot L_{24}}{L_{14} \cdot L_{23}}.$$

§ 8. The easiest and most accurate way of determining the geometrical properties of a given crystal, its axial ratios, and the indices of all its faces is by measuring the angles between its faces. To do this we use an instrument called a *goniometer* (Fig. 9). It usually consists of a fixed collimator C with an illuminated slit S; a fixed telescope T, the centre

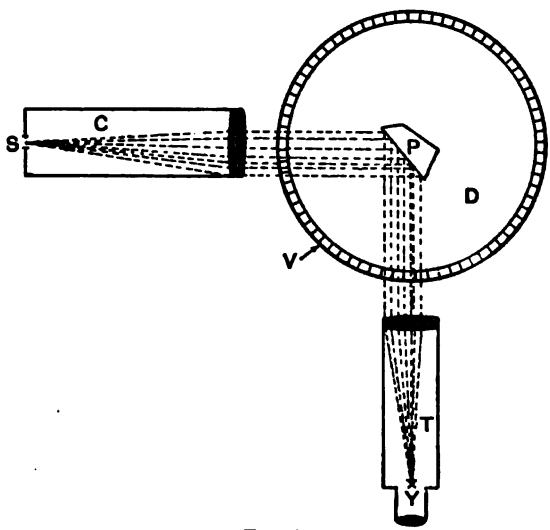

Fig. 9.

of whose field of view is marked with cross wires Y; and a divided circle D attached to a crystal-holder which can be rotated about an axis a perpendicular to the plane of the divided circle, and passing through the centre of this circle. The axes of the collimator and telescope intersect at a point P on the axis a, and are perpendicular to this axis. The crystal is fixed to the holder and is adjusted so that P is its middle point, and so that two of its faces (and therefore the edge in which they meet) are parallel to the axis a. On rotating the crystal about this axis the image of the slit S will be seen in the telescope by reflexion at each of these faces in turn. The circle and crystal are firmly clamped together, and turned till the image of the slit is seen in the centre of the field of

view of the telescope by reflexion at one of the faces. A reading of the circle is then taken by means of a fixed pointer (or vernier) V. The crystal and circle are then turned till the image of the slit is seen by reflexion at the other face, and another reading is taken. The difference between the two readings evidently gives the external angle between the two faces.

The angles between all the faces of one zone can be measured without readjusting the crystal.

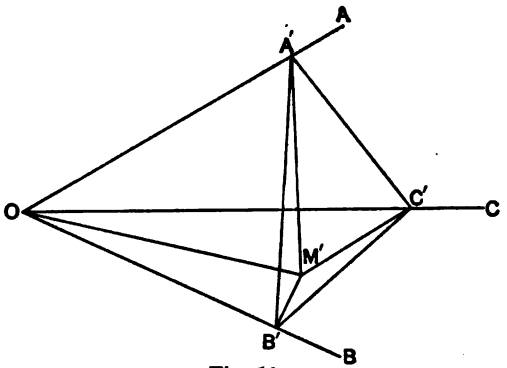

Fig. 10.

§ 9. We are at liberty to choose lines OA, OB, OC parallel to any three edges of a crystal (no two of which are parallel) as crystallographic axes, and the indices of any face (m) not parallel to OA, OB or OC as (111). The indices of faces parallel to OBC, OCA, OAB are then (100), (010), (001) respectively and m is the 'parametral face.'

We now proceed to show how the axial ratios, and the indices of the faces may be calculated from observations (made with the goniometer) of the angles between various faces.

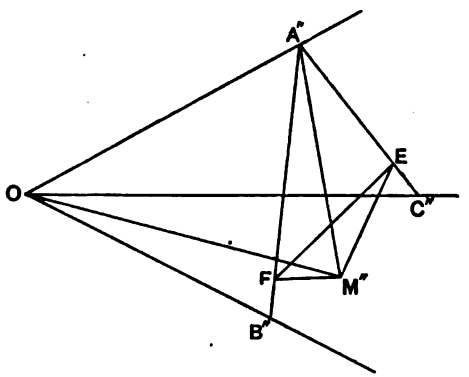

Fig. 11.

Let any plane parallel to m, cut OA, OB, OC and the normal to m through O, in A', B', C', M' (Fig. 10), then $a:b:c$ (the axial ratios)

$$= OA' : OB' : OC' = \frac{OA'}{OM'} = \frac{OB'}{OM'} = \frac{OC'}{OM'}$$

$= \sec M'OA' : \sec M'OB' : \sec M'OC'$, for the angles $OM'A'$, $OM'B'$, $OM'C'$ are right angles.

Again, let any plane perpendicular to OA meet OA, OB, OC and the line through O normal to m in A'', B'', C'', and M'' (Fig. 11). Draw $M''E$, $M''F$ perpendicular to $A''C''$ and $A''B''$ respectively.

Then because OA'' is perpendicular to the plane $A''EM''F$, therefore OA'' is perpendicular to $M''E$, $M''F$.

Hence $M''E$ is perpendicular to $A''C''$ and OA'', and is therefore perpendicular to the plane $OA''C''$; similarly $M''F$ is perpendicular to the plane $OA''B''$.

Let the normal to m make angles λ, μ, ν with the normals to OBC, OCA, OAB; these angles are measured with the goniometer.

Then $OM''E = \pi - \mu$, $OM''F = \pi - \nu$.

Again, let the normals to OBC, OCA, OAB make angles A, B, C* with each other; then evidently

$$\pi - FA''E = FM''E = A.$$

Since A'', F, M'', E lie on a circle of which $A''M''$ is a diameter, therefore $\dfrac{A''M''}{OM''} = \dfrac{FE}{\sin A} \cdot \dfrac{1}{OM''}$

$$= \sqrt{\frac{M''E^2 + M''F^2 - 2\,M''E \,.\, M''F \,.\cos A}{\sin^2 A \,.\, OM''^2}}.$$

Hence, since $OA''M''$, OEM'', OFM'' are right angles,

$$\sin A''OM'' = \sqrt{\frac{\cos^2\mu + \cos^2\nu - 2\cos\mu \,.\cos\nu \,.\cos A}{\sin^2 A}},$$

and consequently $\sec A''OM'' = \sec A'OM'$ (Fig. 10)

$$= \frac{\sin A}{\sqrt{\sin^2 A - \cos^2\mu - \cos^2\nu + 2\cos\mu \,.\cos\nu \,.\cos A}}.$$

Therefore $a : b : c$

$$= \frac{\sin A}{\sqrt{\sin^2 A - \cos^2\mu - \cos^2\nu + 2\cos\mu \,.\cos\nu \,.\cos A}}$$

$$: \frac{\sin B}{\sqrt{\sin^2 B - \cos^2\nu - \cos^2\lambda + 2\cos\nu \,.\cos\lambda \,.\cos B}}$$

$$: \frac{\sin C}{\sqrt{\sin^2 C - \cos^2\lambda - \cos^2\mu + 2\cos\lambda \,.\cos\mu \,.\cos C}}.$$

Let a sphere cut the lines OA, OB, OC and the normal through O to m in A, B, C, M (Fig. 12).

* The angles λ, μ, ν, A, B, C are the external angles of the tetrahedron formed by OBC, OCA, OAB, and m; i. e. the angles actually measured by the goniometer.

The point M is called the *pole* of the face m.

The angles CAB, ABC, BCA of the spherical triangle ABC are the supplements of the angles A, B, C, which the planes OBC, OCA, OAB make with one another; hence the angles between the crystallographic axes are found from the formulae

$$\cos BOC = \frac{\cos B \cdot \cos C - \cos A}{\sin B \cdot \sin C}, \text{ \&c.}$$

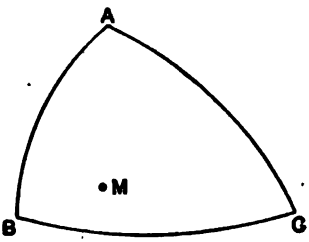

Fig. 12.

The angles MA, MB, MC ($A'OM'$, $B'OM'$, $C'OM'$ of Fig. 10) which determine the axial ratios are not independent. For since the sum of the angles BMC, CMA, AMB is 2π

$$1 + 2\cos BMC \cdot \cos CMA \cdot \cos AMB - \cos^2 BMC - \cos^2 CMA - \cos^2 AMB = 0.$$

But $$\cos BMC = \frac{\cos BC - \cos MB \cdot \cos MC}{\sin MB \cdot \sin MC}, \text{ \&c.,}$$

and $\therefore$ substituting for $\cos BMC$, $\cos CMA$, $\cos AMB$, and reducing, we have

$$1 + 2\cos BC \cdot \cos CA \cdot \cos AB - \cos^2 BC - \cos^2 CA - \cos^2 AB = 2\Sigma\{\cos^2 MA \cdot \sin^2 BC + 2\cos MB \cdot \cos MC(\cos CA \cdot \cos AB - \cos BC)\}.$$

Similarly the angles λ, μ, ν are not independent, for $\pi-\lambda$, $\pi-\mu$, $\pi-\nu$ are the distances of M from the vertices of the polar triangle of ABC (whose sides are A, B, C); and therefore

$$1 + 2\cos A \cdot \cos B \cdot \cos C - \cos^2 A - \cos^2 B - \cos^2 C = \Sigma\{\cos^2\lambda \cdot \sin^2 A + 2\cos\mu \cdot \cos\nu\,(\cos B \cdot \cos C - \cos A)\},$$

or $$\Sigma\sin^2\lambda \cdot \sin^2 A = 2\{1 - \cos A \cdot \cos B \cdot \cos C + \Sigma\cos\mu \cdot \cos\nu\,(\cos B \cdot \cos C - \cos A)\}\ *.$$

This formula is a check on the accuracy of our goniometric measurement of the angles A, B, C, λ, μ, ν.

* This equation gives the relation between the angles which any four planes make with one another. It may be put in the form

$$\begin{vmatrix} 1 & \cos C & \cos B & \cos\lambda \\ \cos C & 1 & \cos A & \cos\mu \\ \cos B & \cos A & 1 & \cos\nu \\ \cos\lambda & \cos\mu & \cos\nu & 1 \end{vmatrix} = 0.$$

See note II at the end of chap. vii.

§ 10. Let m_1 be any other face of the crystal making angles λ_1, μ_1, ν_1, with OBC, OCA, OAB; and let this face meet OA, OB, OC in A_1, B_1, C_1. Then if h, k, l are the indices of m_1

$$h:k:l = \frac{a}{OA_1} : \frac{b}{OB_1} : \frac{c}{OC_1}$$

$$= \sqrt{\frac{\sin^2 A - \cos^2\mu_1 - \cos^2\nu_1 + 2\cos\mu_1 . \cos\nu_1 . \cos A}{\sin^2 A - \cos^2\mu - \cos^2\nu + 2\cos\mu . \cos\nu . \cos A}}$$

$$: \sqrt{\frac{\sin^2 B - \cos^2\nu_1 - \cos^2\lambda_1 + 2\cos\nu_1 . \cos\lambda_1 \cos B}{\sin^2 B - \cos^2\nu - \cos^2\lambda + 2\cos\nu . \cos\lambda . \cos B}}$$

$$: \sqrt{\frac{\sin^2 C - \cos^2\lambda_1 - \cos^2\mu_1 + 2\cos\lambda_1 . \cos\mu_1 \cos C}{\sin^2 C - \cos^2\lambda - \cos^2\mu + 2\cos\lambda . \cos\mu . \cos C}}.$$

This gives us the indices of the face in terms of measured angles.

In practice it is not necessary to use this formula to obtain the indices of every face.

For instance, if five faces n_1, n_2, n_3, n_4, n_5 are such that n_1, n_3, n_5 lie in one zone, and n_2, n_3, n_4 lie in another*, we can find the indices of n_3 when we have found the indices of n_1, n_2, n_4, n_5; for from these latter we can obtain the indices of the two zonal axes, and from them we obtain the indices of n_3 (p. 12).

Again, if four faces lie in a zone and we know the angles between all four faces, and the indices of three of the faces, we obtain the indices of the fourth face by the formula of page 15 (§ 7).

The assemblage of the poles (p. 19) of all the faces of a crystal, or, which is often more convenient, the stereographic projection of this assemblage, may be taken as the geometrical representation of the crystal.

There is no difficulty in drawing this projection for any crystal whose angles have been measured, with the help of chap. i, §§ 4, 5 (p. 6).

The poles of all faces in one zone lie on a great circle of the sphere, and their projections therefore lie on a circle

* Our goniometric observations show us at once which faces lie in a zone; for if the crystal is adjusted so that a zonal axis is parallel to the axis of the instrument and is rotated about this axis, images of the slit can be seen in the telescope by reflexion at faces which belong to the zone, and at such faces only.

which passes through the ends of a diameter of the circle s (cf. Fig. 7, p. 6).

It is outside the purpose of this treatise to pursue this subject further. We have proved all the theorems necessary for the deduction of the geometrical properties of a crystal from goniometric measurements; for the details of the application of these theorems we must refer the reader to treatises on practical crystallography.

NOTE.—The formulae of §§ 9 and 10 can be put in a form suitable for logarithmic calculation by the substitutions

$$\sin^2 A - \cos^2\mu - \cos^2\nu + 2\cos\mu \cdot \cos\nu \cdot \cos A$$
$$= 4 \sin \tfrac{1}{2}(A+\mu+\nu) \cdot \sin \tfrac{1}{2}(-A+\mu+\nu)$$
$$\cdot \sin \tfrac{1}{2}(A-\mu+\nu) \cdot \sin \tfrac{1}{2}(A+\mu-\nu), \text{ etc.}$$

CHAPTER III

SYMMETRY.

§ 1. A figure * is said to be *translated* when each point of it is moved through the same distance in the same direction.

This motion (a *translation*) can be represented by a straight line whose length and direction (but not position) are given.

A figure is said to be *rotated* about a straight line (called the 'axis of rotation') when the figure moves as if rigidly connected with the straight line, every point of which is fixed in space.

The angle between the final and initial positions of any plane parallel to the axis and rigidly connected with the figure is called the angle of rotation.

Each point of the figure evidently describes a circle about some point of the axis as centre.

This motion (a *rotation*) can be represented by a finite straight line (whose length gives the angle of rotation), which lies along the axis, but whose position in the axis is not fixed.

§ 2. Two figures are said to be *congruent* to one another when one can be brought to coincide with the other by a movement which does not alter the relative position of its own parts †.

Two figures which are congruent to the same figure are evidently congruent to one another.

A figure U can be brought into coincidence with any congruent figure U' by a translation followed by a rotation.

For let the point O of U correspond to the point O' of U', and the lines OA, OB of U to the lines OA', OB' of U' (so that when U coincides with U', O coincides with O', OA with OA', OB with OB'), and let U be translated through a distance OO' in a direction parallel to OO' ‡.

* In future we shall mean by 'figure' any collection of points, lines, and planes.

† That is, a movement in which the figure behaves as a rigid body.

‡ Such a movement is represented in magnitude and direction by the straight line OO', and is called 'the translation OO'.'

Then O is brought into coincidence with the corresponding point O'; suppose that OA, OB are brought into the positions OA_1 and OB_1 respectively.

Let a sphere whose centre is O $(= O')$ cut OA_1, OB_1, in A_1, B_1, and the corresponding lines of U' in A' and B'; then evidently the points A_1, B_1 in U correspond to A', B' in U'.

Bisect the great circles A_1A', B_1B' at right angles by great circles meeting at I (Fig. 13).

Then obviously $IA_1 = IA'$; $IB_1 = IB'$; also $A_1B_1 = A'B'$, for A_1, B_1 correspond to A', B'.

Therefore the angle A_1IB_1 = the angle $A'IB'$, and hence the angle A_1IA' = the angle B_1IB'.

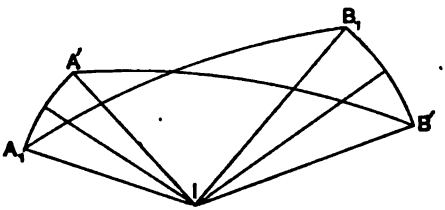

Fig. 13.

Therefore A_1 is brought to A'*, and B_1 to B' by a rotation about OI through an angle A_1IA' $(= B_1IB')$.

Now a figure is completely fixed in position when the positions of three of its points are known.

Hence if three points of the figure U coincide with the corresponding points of a congruent figure U', U will completely coincide with U'.

But O is brought to O', A to A', and B to B' by the translation OO' followed by a rotation about OI; $\therefore$ U is brought to coincide with U' by the translation OO' followed by a rotation about OI.

A figure can be brought into coincidence with any congruent figure by a rotation followed by a translation.

For since U is brought into coincidence with U' by a translation followed by a rotation; $\therefore$ U' is brought into coincidence with U by a rotation followed by a translation.

§ 3. *A figure is rotated about an axis OA through an angle α, and then about an axis OB through an angle β; to find the resulting movement.*

By § 2, since the point O of the figure is fixed, therefore the resulting movement is a rotation about an axis through O.

We shall consider angles positive if they are measured in the clockwise direction.

Let OA, OB meet a sphere whose centre is O in A and B.

* i. e. 'A_1 is brought into coincidence with A'.'

Make the spherical angle $CAB = -\frac{a}{2}$ and the spherical angle $ABC = -\frac{\beta}{2}$.

Produce the great circle BC to W, draw the great circle AWE perpendicular to BCW, and take the point E such that $AW = WE$ (Fig. 14).

Let $C'AB = \frac{a}{2}$ and $ABC' = \frac{\beta}{2}$.

Then C is brought to C' by the rotation about OA through the angle a; and C' is brought to C by the rotation about OB through an angle β.

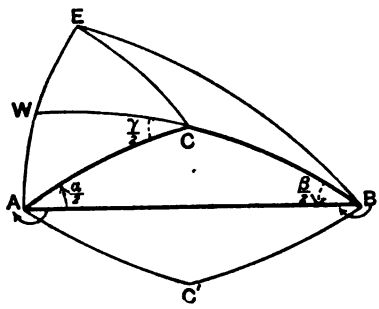

Fig. 14.

Again A is left unmoved by the rotation about OA and is brought to E by the rotation about OB.

Hence C is left unmoved by the resultant of the rotations first about OA and then about OB, while A is brought to E.

Therefore the resultant motion (which is some rotation about an axis through O) must be a rotation about OC through an angle $ECA = 2\,WCA$ in the positive (clockwise) direction.

The above is called Euler's or Rodrigues' construction.

From the spherical triangle ABC we have, if ϕ is the angle between OA and OB,

$$\cos\frac{a}{2}\cdot\cos\frac{\beta}{2} - \sin\frac{a}{2}\cdot\sin\frac{\beta}{2}\cdot\cos\phi = \cos ACW = \cos\frac{\gamma}{2},$$

if γ is the angle of rotation about OC; and

$$\therefore\ \tfrac{1}{2}\{(\cos a+\cos\beta-1)\sin^2\phi+\cos a\cdot\cos\beta\cdot(1+\cos^2\phi) - 2\sin a\cdot\sin\beta\cdot\cos\phi\} = \cos\gamma.$$

As a particular case we have $\phi = 0$; the axes OA, OB are then parallel and O is at infinity. The same geometrical construction holds for the position of the axis of the resultant rotation. This axis is now parallel to OA and OB, and $\gamma = a+\beta$.

The resultant of a rotation first about OB and then about OA is easily seen to be a rotation through the same angle γ in the same direction about the axis OC'.

As another particular case take $a = \beta = \pi$.

Then OC is perpendicular to the plane OAB and

$$\cos\frac{\gamma}{2} = -\cos\phi. \qquad \therefore\ \frac{\gamma}{2} = \pi - \phi.$$

If $\phi = \frac{\pi}{2}$; $\alpha = \beta = \gamma = \pi$.

§ 4. If from any point O of a figure U, a perpendicular ON is drawn to a fixed plane q, and produced to O' such that OO' is bisected at N; then U', the locus of O', is called the *reflexion* of U in the plane q.

If U' is the reflexion of U, evidently U is the reflexion of U' in the same plane q.

If C is a fixed point, and O any point of a figure U; and if OC is produced to O'' so that OO'' is bisected at C; then U'', the locus of O'', is called the *inverse* of U about the point C*.

If U'' is the inverse of U, evidently U is the inverse of U'' about the same point.

If the point C lies in the plane q; then U'' can be brought into coincidence with U' by a rotation through π about an axis through C perpendicular to q.

For this axis being perpendicular to q, is parallel to OO', and lies therefore in the plane $O''OO'$; let it meet $O'O''$ in K (Fig. 15).

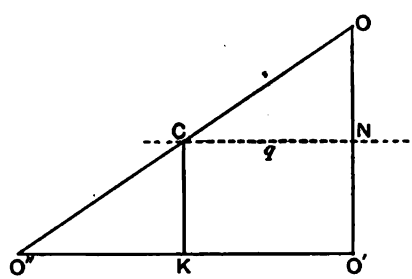

Fig. 15.

Then $\because\ ON = NO'$ and $OC = CO''$, therefore $O'O''$ is parallel to CN; and therefore, since CK is parallel to OO', and $O''C = CO$, the triangles $O''CK$, CON are equal in all respects.

$\therefore\ O''K = CN = KO'$ (for $CNO'K$ is a parallelogram), and the angle $CKO'' = ONC =$ a right angle.

Hence O'' is brought to O' by a rotation through π about CK; and this being true whatever point O of U we start with, must be true for all the points of U' and U'', and therefore for the whole figures. Evidently U' is brought to U'' † by the same rotation.

* Distinguish from the 'inverse with respect to a point'; cf. chap. i, § 2, p. 1.

† i.e. 'the figure U' is brought into coincidence with U''' and so in future.

The theorem is also readily proved by consideration of Fig. 5 (p. 4); it may be stated in the form:

If U'' is the inverse of U about a point C; and if U'' be brought to U' by rotation through two right angles about any axis through C; then U' is the reflexion of U in a plane through C at right angles to the axis.

§ 5. *If U' is the reflexion of a figure U in a plane q_1 and U'' is the reflexion of U in a plane q_2; then U' can be brought to coincide with U'' by a rotation through an angle 2α about the intersection of q_1 and q_2, where α is the angle between the planes q_1 and q_2.*

For let O' be the reflexion in q_1 of any point O of U; and O'' the reflexion of O in q_2 (Fig. 16).

Let OO' meet q_1 in M, and let OO'' meet q_2 in N.

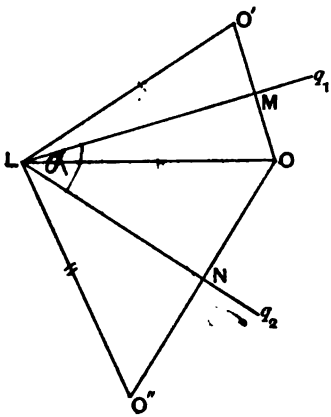

Fig. 16.

Then the plane $O'OO''$ is perpendicular to q_1 and q_2, and therefore cuts their line of intersection at right angles (in L say).

Then because LM bisects OO' at right angles, therefore $OL = O'L$ and the angle $OLO' = 2 . OLM$.

Similarly $OL = O''L$ and

$$O''LO = 2 . NLO.$$

Therefore $O'L = O''L$ and

$$O''LO + OLO' = 2(NLO + OLM),$$

or $O''LO' = 2 . NLM = 2\alpha$.

Hence O' is brought to O'' by a rotation through 2α about the intersection of q_1 and q_2; and this being true whatever point O of U we started from, it is true for the whole figures U, U', U''.

Corollary 1. Two figures which are reflexions of the same figure are congruent to one another.

Corollary 2. If U' is the reflexion of any figure U in a plane q, and if U' is brought to coincidence with U'' by a rotation through an angle 2α about an axis l lying in q, then U'' is the reflexion of U in a plane through l which makes an angle α with q.

§ 6. Two figures are said to be *enantiomorphous* when one is the reflexion of the other in any plane, or when one can be brought by a movement which leaves the relative positions of

its parts unaltered into such a position that it is the reflexion of the other in a plane.

A figure and its inverse about a point are enantiomorphous by § 4.

Two figures U' and U'' which are enantiomorphous to the same figure U, are congruent to one another. For by definition U' is congruent to some figure U_1' which is the reflexion of U in some plane, and U'' is congruent to some figure U_1'' which is the reflexion of U in some other plane. Now by the previous section U_1' and U_1'' are congruent, hence U', U_1'' being both congruent to U_1' are congruent to one another; and $\therefore$ U' and U'' being both congruent to U_1'' are congruent to one another.

A figure U can be brought into such a position that it is the reflexion of any enantiomorphous figure U' in a plane, by a rotation about an axis perpendicular to the plane.

For let O, O' be corresponding points in U and U'. Bisect OO' in E; let U'' be the inverse of U' about E; then U and U'' being both enantiomorphous to U' are congruent to one another. Also the point O of U'' corresponds to O' of U' and therefore to O of U; i.e. U and U'' have two corresponding points (one in each figure) identical. Hence U can be brought to coincide with U'' by a rotation about some axis l passing through O. Let m be the line through E parallel to l. Then by the end of § 4, if U'' be brought to U''' by a rotation through π about the axis m, U''' is the reflexion of U' in a plane through E perpendicular to l or m.

Now U is brought to U''' by two consecutive rotations about the parallel axes l, m; and these rotations are by § 3 equivalent to some single rotation about an axis n parallel to l*. But U''' is the reflexion of U' in a plane perpendicular to l or n; hence U can be brought by a rotation about an axis n into coincidence with the reflexion of U' in a plane perpendicular to n.

We may put the results of this section in the following form:

*A figure may be brought to coincidence with any enantiomorphous figure by a rotation about an axis followed by reflexion in a plane perpendicular to that axis**.

This combination of a rotation about an axis through an

* If the angle of rotation about l is π, n is at infinity and the angle of rotation about n is infinitesimal; that is, U is brought to U''' by a translation perpendicular to l, or U is brought to coincidence with U' by a translation followed by reflexion in a plane parallel to the direction of the translation.

angle ω followed by a reflexion in a plane perpendicular to that axis, is called a *rotatory-reflexion* * of angle ω.

Since U is brought to U' by a rotation followed by a reflexion, $\therefore$ U' is brought to U by a reflexion followed by a rotation; hence:

A figure may be brought to coincidence with any enantiomorphous figure by a reflexion in a plane followed by a rotation about an axis perpendicular to that plane.

Any two enantiomorphous figures have two corresponding points (one in each) common; for the point where the axis n meets the perpendicular plane through E corresponds to itself in the two figures †.

§ 7. A translation, rotation, reflexion or the result of combining any of these is called an *operation* ‡. An operation acting on a figure must bring it into coincidence, either with a congruent figure, when the operation is said to be *of the first sort*; or with an enantiomorphous figure, when the operation is said to be *of the second sort*. An operation of the first or second sort is also called a *movement*. It should be noticed that the movements with which we are concerned, are those which do not alter the distance between any two given points.

An operation may conveniently be represented by a symbol; thus a rotation about an axis a through an angle α may be represented by A or $A(\alpha)$; a reflexion by S, an inversion about a point by I, &c.

If two or more operations are written after one another it means that the operations are applied to the figure under consideration in order from left to right.

Thus $A.B.C$ means that first the operation A acts, then the operation B, and finally the operation C; the resulting movement is called the 'product' or 'resultant' of A, B, and C.

If in § 8 (p. 24) $A(\alpha)$ is the rotation about OA, $B(\beta)$ that about OB, $C(\gamma)$ that about OC; we may write the results there arrived at in the form:

$$A(\alpha).B(\beta) = C(\gamma), \text{ or } A.B = C;\ B(\beta).A(\alpha) = C'(\gamma), \text{ or } B.A = C'$$

(cf. Fig. 14); where the sign of equality (=) denotes identity of effect on *any* figure.

* German—Drehspiegelung.

† In the case mentioned in the footnote on p. 27 this point is at infinity.

‡ This word is usually used in a far wider sense; but translations, rotations, reflexions, and combinations of these are the only kinds of 'operation' considered in this book.

If A, B are two operations such that $A \,.\, B = B \,.\, A$, A and B are said to be *permutable*; from what has just been said we see that two rotations about intersecting axes are not in general permutable. If it is desired to specify the figure on which any operation or series of operations acts, some symbol denoting the figure is placed in square brackets before the symbol or symbols denoting the operation or series of operations.

If a figure remains fixed we say that *the identical operation* acts on it; the identical operation is denoted by the symbol 1. We have evidently $1 \,.\, A = A \,.\, 1 = A$, where A is any operation. The operation $A \,.\, A \,.\, A \,.\,...$ to n factors is written A^n.

The reverse operation to A is written A^{-1}, so that

$$A \,.\, A^{-1} = A^{-1} \,.\, A = 1,$$

$A^{-1} \,.\, A^{-1} \,.\, A^{-1} \,.\,...$ to n factors is written A^{-n}.

If A, B, C are any operations, and $A \,.\, B = P$, $B \,.\, C = Q$, then $P \,.\, C = A \,.\, Q$.

For let U be any figure, and let

$$[U]A = [U']\,1, [U']B = [U'']\,1, [U'']\,C = [U''']\,1;$$

that is, let A bring U to U', B bring U' to U'', and C bring U'' to U'''.

Then $[U]\,P = [U]\,A \,.\, B = [U']\,B = [U'']\,1$, and therefore $[U]\,P \,.\, C = [U''']\,1$.

Similarly $[U']\,Q = [U']\,B \,.\, C = [U'']\,C = [U''']\,1$, and therefore $[U]\,A \,.\, Q = [U''']\,1$.

Hence since U is *any* figure $P \,.\, C = A \,.\, Q$.

This result may be written $(A \,.\, B) \,.\, C = A \,.\, (B \,.\, C)$; it may readily be extended, and we see that in general operations of the kind we are considering obey the associative law though not the commutative.

We deduce from this that

$$A^{-2} \,.\, A^2 = A^{-1} \,.\, (A^{-1} \,.\, A) \,.\, A = A^{-1} \,.\, A = 1.$$
$$A^{-3} \,.\, A^3 = A^{-1} \,.\, (A^{-2} \,.\, A^2) \,.\, A = A^{-1} \,.\, A = 1,$$

and in general $A^{-n} \,.\, A^n = 1$.

Similarly $A^n \,.\, A^{-n} = 1$.

It readily follows that $A^p \,.\, A^q = A^{p+q}$, where p and q are any positive or negative integers; if by A^0 we mean the identical operation.

If C and D are equivalent operations so that $C = D$; then $C \,.\, A = D \,.\, A$ and $A \,.\, C = A \,.\, D$, where A is any operation.

For let U be any figure, and let $[U]\,C = [U']\,1$, $[U']\,A = [U'']\,1$,

Then because $C = D$, therefore $[U]\,D = [U']\,1$.

Hence $[U]\,C\,.\,A = [U']\,A = [U'']\,1$, and $[U]\,D\,.\,A = [U']\,A = [U'']\,1$; and therefore $C\,.\,A$ and $D\,.\,A$ both bring U to U'' so that $C\,.\,A = D\,.\,A$.

We may prove $A\,.\,C = A\,.\,D$ in a similar way.

Hence *we may multiply both sides of an equation of operations either on the right or on the left by the same operation.*

Similarly, if $C\,.\,A = D\,.\,A$.

Then $C\,.\,A\,.\,A^{-1} = D\,.\,A\,.\,A^{-1}$ by what we have just proved and $\therefore\ C = D$.

Again, if $A\,.\,C = A\,.\,D$ then $A^{-1}\,.\,A\,.\,C = A^{-1}\,.\,A\,.\,D$ and $\therefore\ C = D$.

Hence *we may divide both sides of an equation of operations either on the right or on the left by the same operation.*

§ 8. *The product of two operations is an operation of the first or second sort according as the operations are of the same or of different sorts.*

Let L, M be any two operations, and let L bring any figure U to U', and let M bring U' to U''.

First suppose L and M of the same sort.

Then U and U'' are either both congruent to U' or both enantiomorphous to U'; and hence U and U'' are congruent.

But $L\,.\,M$ brings U to U''; therefore $L\,.\,M$ is an operation of the first sort.

Next let L, M be two operations of different sorts; in this case either U' is congruent to U and not to U'', or to U'' and not to U. Hence U and U'' are enantiomorphous; and therefore, since $L\,.\,M$ brings U to U'', $L\,.\,M$ is an operation of the second sort.

We may readily extend the above and obtain the theorem :—

The product of any number of operations is an operation of the first or second sort, according as the number of operations of the second sort in the product is even or odd.

§ 9. If $A(\alpha)$, $A(\beta)$, $A(\gamma)$, ... are rotations through $\alpha, \beta, \gamma, \ldots$ about the same axis, we have evidently $A(\alpha)\,.\,A(\beta)\,.\,A(\gamma)\ \ldots = A(\alpha + \beta + \gamma + \ldots)$.

In particular $A^n(\alpha)$* $= A(n\alpha)$.

Again, since $A(\alpha)\,.\,A(-\alpha) = A(0) = 1$.

$\therefore\ A(-\alpha) = A^{-1}(\alpha)$.

* We shall denote $\{A(\alpha)\}^n$ by $A^n(\alpha)$ in future.

§ 10. If $A(a)$ be a rotation through a about an axis a, and S a reflexion in a plane perpendicular to a, *then* $A(a)$ *and* S *are permutable operations.*

The point O in which the axis and plane meet is evidently left unmoved by either operation.

Let P' be any point; on a plane perpendicular to a form the stereographic projection of the sphere whose centre is O and radius OP'.

Then (using the notation of p. 4), $P(+)$, the projection of P', is brought by $A(a)$ to $P_1(+)$*, (Fig. 17), where $P_1CP = a$, and $CP = CP_1$; and $P_1(+)$ is brought to $P_1(o)$ by S.

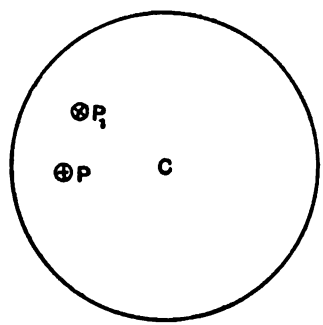

Fig. 17.

Therefore $A(a).S$ brings $P(+)$ to $P_1(o)$.

Again S brings $P(+)$ to $P(o)$, and $A(a)$ brings $P(o)$ to $P_1(o)$; therefore $S.A(a)$ brings $P(+)$ to $P_1(o)$.

Hence, since P' is any point, $A(a).S = S.A(a)$. As a particular case $A(\pi).S = S.A(\pi) = I$, the inversion about O.

$A(a).S = S.A(a)$ is the general operation of the second sort (rotatory-reflexion).

§ 11. Let U be any figure, and $A'(a) = A(a).S$ be this general operation of the second sort; we shall consider the properties of powers of A'.

First let $a = o$.

If U is any figure and $[U]S = [U']1$; then we know that $[U']S = [U]1$†. Therefore $[U]S^2 = [U']S = [U]1$, and hence $S^2 = 1$.

It follows that $S^{2m} = (S^2)^m = 1^m = 1$, and that

$$S^{2m+1} = 1.S = S;$$

where m is any integer.

Next let $a = \pi$. Then $I^2 = A(\pi).S.S.A(\pi) = A(\pi).A(\pi) = A(2\pi) = 1$.

It follows that $I^{2m} = 1$, and that $I^{2m+1} = I$.

Now suppose that a is not equal to o or a multiple of π.

Then $A'^2(a) = A.S.A.S = A.A.S.S = A^2(a) = A(2a)$; and $A'^3(a) = A(2a).A.S = A(3a).S$.

Similarly $A'^4(a) = A(4a)$; $A'^5(a) = A(5a).S$, &c.

* i. e. the point of which $P(+)$ is the projection is brought into coincidence with the point of which $P_1(+)$ is the projection.

† That is, 'if U' is the reflexion of U in any plane, U is the reflexion of U' in the same plane' (p. 25).

Hence the operations

$$\left.\begin{matrix} 1, & A'^2, & A'^4, & A'^6, & \dots \\ A', & A'^3, & A'^5, & A'^7, & \dots \end{matrix}\right\}$$

are equivalent to

$$\left.\begin{matrix} 1, & A(2a), & A(4a), & A(6a), & \dots \\ A(a).S, & A(3a).S, & A(5a).S, & A(7a).S, & \dots \end{matrix}\right\}.$$

§ 12. If U, U' be any two congruent or enantiomorphous figures then U can be brought into coincidence with U' by a rotation, a translation, a combination of rotation and translation, or a rotatory-reflexion.

Now *if U' is identical with U*, U is brought into coincidence with itself by some operation of the kind just described. In that case U is said to have *symmetry*, and the operation which brings U to self-coincidence is called a *symmetry-operation* of U. It may be possible to bring U to self-coincidence * in several different ways, that is, by means of several different symmetry-operations; but in every case the operation must be equivalent to a rotation, a translation, a combination of rotation and translation, or a rotatory-reflexion; hence:

The only kinds of symmetry a figure can possess are those due to a rotation, a translation, a combination of rotation and translation, or a rotatory-reflexion.

In this statement a simple reflexion, a translation followed by a reflexion in a parallel plane, and inversion about a point are included as special cases of a rotatory-reflexion.

If a figure U is brought to self-coincidence by a rotation about any line, that line is called a *rotation-axis*, 'axis of symmetry of the first sort,' or 'axis of symmetry' of U.

If a figure U coincides with its own reflexion in a plane, that plane is called a *symmetry-plane* or 'plane of symmetry' of U.

If a figure U coincides with its own inverse about a point, that point is called a *centre of symmetry* or 'centre of inversion' of U.

* When we say that a figure U is brought to self-coincidence, we do not mean that each point, line, or plane of U is brought to coincide with itself; but that each point, line, or plane of U is brought into the position previously occupied either by itself or by some other point, line, or plane of U.

For example: if the cube $AA_1A'A_1'A''A_1''A'''A_1'''$ (Fig. 126) is rotated through $\frac{2\pi}{3}$ about the diagonal AA_1, the line AA_1 does not alter its position, while the line $A'A_1'$ is brought into the position previously occupied by $A''A_1''$, the line A_1A' into the position previously occupied by A_1A'', &c.; but the cube as a whole is brought into self-coincidence by the rotation.

If a figure U is brought to self-coincidence by a rotatory-reflexion about an axis, that axis is called a *symmetry-axis of the second sort* or 'symmetry-axis of rotatory-reflexion' of U.

Such axes, planes, &c. are called collectively *symmetry-elements* or 'elements of symmetry' of U.

The identical operation is by definition a symmetry-operation of *any* figure.

§ 13. Let $A, B, C, \ldots, L$ be symmetry-operations of U, so that $[U]A = [U]B = [U]C = \ldots = [U]L = [U]1$. Then $[U]A.B.C.\ldots.L = [U]B.C.\ldots.L = [U]C.\ldots.L = \ldots = [U]L = [U]1$.

Hence *if $A, B, C, \ldots, L$ are symmetry-operations of any figure, $A.B.C.\ldots.L$ is also a symmetry-operation.*

Again, if H is a symmetry-operation of U,

$$[U]H^{-1} = [U]H.H^{-1} = [U]1;$$ hence:—

If H is a symmetry-operation of any figure so is H^{-1}.

Combining the two previous theorems we have:

If $A, B, C, \ldots, L$ are symmetry-operations of any figure, $A^p.B^q.C^r.\ldots.L^z$ is also a symmetry-operation; where $p, q, r, \ldots, z$ are any positive or negative integers.

A particular case of this is the following theorem:

If A is a symmetry-operation of any figure so is A^n; where n is any positive or negative integer.

For example; if the rotations $A(\alpha), A(\beta), A(\gamma), \ldots$ about the same axis are symmetry-operations of any figure; so are $A(n\alpha), A(n\beta), A(n\gamma), \ldots$ and in general $A(p\alpha + q\beta + r\gamma + \ldots)$; where $n, p, q, r, \ldots$ are any positive or negative integers.

§ 14. Let α be the smallest positive angle through which a figure U must be turned about a rotation-axis a to bring it to self-coincidence; then, if α is not infinitesimal, 2π is a multiple of α.

For if $A(\alpha)$ represent a rotation through α about a, then evidently $A(\alpha)$ is identical with $A(\alpha + 2m\pi)$ where m is a positive or negative integer.

Now if 2π is not a multiple of α let 2π lie between $r\alpha$ and $\overline{r+1}\alpha$, where r is a positive integer; then $\alpha > \overline{r+1}\alpha - 2\pi > 0$. But since $A(\alpha)$ brings U into self-coincidence, therefore $A^{r+1}(\alpha) = A(\overline{r+1}\alpha) = A(\overline{r+1}\alpha - 2\pi)$ does so. Hence

a rotation $A\ (\overline{r+1}\,a-2\pi)$ about a whose angle is positi and $< a$ brings U to self-coincidence, contrary to hypothesis

Let $2\pi = na$, or $a = \frac{2\pi}{n}$; where n is a positive intege then the axis is called a digonal, trigonal, tetragonal, pent gonal, hexagonal, ... or 2-al, 3-al, 4-al, 5-al, 6-al, ... axis of sy metry according as $n = 2, 3, 4, 5, 6, \ldots$.

The only distinct symmetry-operations of U due to rotatio about a are $1, A, A^2, \ldots, A^{n-1}$; or $1, A(a), A(2a), \ldots, A(\overline{n-1}$

For evidently $A^{(rn+s)}$ is equivalent to A^s and $A^{(rn-s)}$ A^{n-s} (r being any integer, and s zero or a positive integer $< n$ so that every power of A is equivalent to one or other of t series $1, A, A^2, \ldots, A^{n-1}$.

Again if $A(\omega)$ is a symmetry-operation of U, ω is a mu tiple of a. For otherwise it would be possible to find a integer p such that $pa + \omega$ is a positive angle $< a$ a > 0; and in that case $A(pa + \omega)$ would be a rotation throu an angle $< a$ and > 0 bringing U to self-coincidence, contra to hypothesis.

§ 15. Now let $A' = A(a).S$ be a symmetry-operation of figure U (using the notation of p. 31), where a is positi and $\neq 0$ or π, and suppose that no rotatory-reflexion $A(\gamma)$. about the same axis and plane is a symmetry-operation of if γ lies between 0 and a exclusive.

Let β be the smallest positive angle (zero excluded) f which the rotation $A(\beta)$ is a symmetry-operation of U; the since $A'^2 = A(2a)$ is a symmetry-operation of U, $\beta \geqslant 2a$.

Now since $A(a).S$, $A(-2a)$, $A(\beta)$, $A(-\beta)$ are symmetry operations of U, so are $A(-\beta).A(a).S = A(a-\beta).S$, an $A(\beta).A(-2a).A(a).S = A(\beta - a).S$.

Hence $\beta \sim a$ is either $= 0$ or else $\nless a$. But $\beta \geqslant 2a$ an $\neq 0$, therefore $\beta = a$ or $2a$.

Now 2π is a multiple of β (p. 33), therefore 2π is multiple of a.

Hence if $na = 2\pi$, n is a positive integer; the axis of th rotatory-reflexion is then called an n-al *symmetry-axis of th second sort*.

First suppose n odd.

In this case $(A'^2)^n = A^n(2a) = 1$;

Hence evidently the only distinct powers of A' are

$$\left\{\begin{matrix} 1, & A(2a), & A(4a), & \ldots, & A(\overline{2n-2}\,a) \\ A(a).S, & A(3a).S, & A(5a).S, & \ldots, & A(\overline{2n-1}\,a) \end{matrix}\right\}$$

(cf. p. 32).

Since $A(n+r)a = A(2\pi + ra) = A(ra)$, these are equivalent to

$$\left\{\begin{array}{llllll} 1, & A(2a), & A(4a), & \ldots, A(\overline{n-3}\,a), & A(\overline{n-1}\,a), & \\ A(a).S, & A(3a).S, & A(5a).S, & \ldots, A(\overline{n-2}\,a).S, & S, & \\ & & A(a), & A(3a), & \ldots, A(\overline{n-2}\,a) & \\ & & A(2a).S, & A(4a).S, & \ldots, A(\overline{n-1}\,a) & \end{array}\right\}$$

or (rearranging) to

$$\left\{\begin{array}{llll} 1, A(a), & A(2a), & A(3a), & \ldots, A(\overline{n-1}\,a) \\ S, A(a).S, & A(2a).S, & A(3a).S, & \ldots, A(\overline{n-1}\,a).S \end{array}\right\} \ldots \text{(i)}.$$

It is readily proved that these are the only operations, whether rotations or rotatory-reflexions, about the axis of $A(a).S$, which are symmetry-operations of U.

Next suppose n even.

In this case $(A'^2)^{\frac{n}{2}} = 1$; hence evidently the only distinct powers of A' are

$$\left\{\begin{array}{llll} 1, & A(2a), & A(4a), & \ldots, A(\overline{n-2}\,a) \\ A(a).S, & A(3a).S, & A(5a)S, & \ldots, A(\overline{n-1}\,a) \end{array}\right\} \ldots \text{(ii)}.$$

If n is a multiple of 4, we use this arrangement; if however n is a multiple of 2 but not of 4 we arrange the operations in the second line of (ii) in the order

$$A\left(\tfrac{n}{2}a\right).S,\ A\left(\tfrac{n+4}{2}a\right)S,\ \ldots,\ A(\overline{n-1}\,a).S,\ A(a).S,\ A(3a).S, \ldots, A\left(\tfrac{n-4}{2}a\right).S.$$

If an inversion about the intersection of the axis and plane of the rotatory-reflexion be denoted by $I = A(\pi).S$, these are equivalent to $I,\ A(2a).I,\ A(4a).I,\ \ldots, A(\overline{n-2}\,a).I$,

since $\dfrac{na}{2} = \pi$, and $A(2\pi) = 1$.

(ii) may then be written

$$\left\{\begin{array}{llll} 1, & A(2a), & A(4a), & \ldots, A(\overline{n-2}\,a) \\ I, & A(2a).I, & A(4a).I, & \ldots, A(\overline{n-2}\,a).I \end{array}\right\} \ldots \text{(iii)}.$$

We may put the results of this section in the following form.

Let a line a meet a perpendicular plane σ in the point O.

Then if a is an n-al symmetry-axis of the second sort of any figure U, and σ is the corresponding plane, n being *odd*; a is an n-al rotation-axis of U and σ is a symmetry-plane.

Conversely if a is an n-al rotation-axis, n being *odd*, and σ is a symmetry-plane, then a is an n-al symmetry-axis of the second sort.

Again if a is an n-al symmetry-axis of the second sort a· σ the corresponding plane, n being *even*, then a is a $\frac{n}{2}$-al ro· tion-axis; and if n is even but $\frac{n}{2}$ odd, O is a centre symmetry.

Conversely if a is an $\frac{n}{2}$-al rotation-axis, $\frac{n}{2}$ being an o integer, and O is a centre of symmetry, then a is an n symmetry-axis of the second sort.

§ 16. The normals from a fixed point O within a crystal the crystal faces form a figure, which is characteristic of t substance. This figure is often found to possess symmet Similarly if radii vectores are drawn from O to repres· any physical property of the crystal in different directic about O, the figure consisting of these vectors has also oft elements of symmetry. The symmetry of all the figu formed by using different physical properties and the natu growth of faces is not always the same; the crystal is s· to possess those elements of symmetry, and those on which are common to *all* such figures.

It is found that usually the system of radii vectores rep senting any physical property has all the symmetry-eleme· possessed by the system of normals through a point to 1 faces. We shall therefore for the present (while we : dealing with crystal symmetry) mean by the word 'cryst· either (1) a collection of finite lines drawn through any poin· perpendicular to the faces, only those lines perpendicular faces whose physical properties—lustre, hardness, &c.—are · same being of equal length, or (2) a collection of pla· parallel to the faces, those planes parallel to faces wh physical properties are the same being equidistant from O, (3) the finite closed polyhedron formed by such planes.

It is indifferent whether we take the collection of lines, · collection of planes, or the polyhedron, as the 'crystal,' wl we are only concerned with the symmetry; for any operat which brings one of them to self-coincidence obviously bri· the other two also to self-coincidence. When, however, are dealing with properties of crystal edges and faces polyhedron is the most convenient representation, for surfaces and edges are parallel to the actual faces and ed of the crystal as found in nature *.

* In any crystal found in nature faces whose physical properties are same are not in general equidistant from any point inside the crystal, ow

The point O (which may be chosen at will) is called the *centre* of the crystal; it is evidently left unmoved by any operation which brings the 'crystal' to self-coincidence; hence:

Any symmetry-operation of a crystal leaves its centre unmoved.

It follows from this that the only symmetry-operations which a crystal can have are rotations about axes passing through its centre, reflexions in planes through its centre, rotatory-reflexions whose axes meet the corresponding planes at the centre, and an inversion about the centre.

§ 17. *Any symmetry-plane of a crystal is parallel to a possible* crystal face, and a line perpendicular to it is parallel to a possible crystal edge.*

For let p, q be planes parallel to two possible faces, and let p', q' be their reflexions in the symmetry-plane (σ). Then p', q' are also parallel to crystal faces, since σ is a symmetry-plane of the crystal. But the intersections of p, p' and of q, q' lie in σ; hence σ contains lines parallel to two possible edges and is therefore parallel to a possible crystal face.

Again let u, v be parallels through O to any possible edges, and u', v' their reflexions in σ; then the plane through u, u' and the plane through v, v' are parallel to possible faces and they meet in a perpendicular to σ, which is therefore parallel to a possible edge.

Corollary. σ contains an indefinite number of lines parallel to possible edges; for the intersection of σ with a plane parallel to *any* possible face is parallel to a possible edge.

§ 18. *An axis of symmetry is parallel to a possible crystal edge, and the plane perpendicular to it is parallel to a possible crystal face.*

Case (1). n-al axes of the first sort†; n even.

(*a*). $n = 2$. Let OB be the axis of symmetry and Oe, Of parallels to any two possible crystal edges. Let Oe', Of' be the positions to which Oe, Of would be brought by a rotation through π about OB (Fig. 18).

Then Oe', Of' are parallel to possible edges; and hence the planes Oee', Off' are parallel to possible faces; therefore their line of intersection OB is parallel to a possible edge.

to extraneous disturbances during the growth of the crystal; but we may assume that, had the growth been undisturbed, such faces would be all equidistant from some internal point.

* Cf. end of § 5, chap. ii, p. 18.

† i. e. rotation-axes.

Again the planes Oef, $Oe'f'$ are parallel to possible fac and meet in a line perpendicular to OB which is parallel a possible edge.

Similarly the planes Oef', $Oe'f$ are parallel to possible fac and meet in a line pe pendicular to OB whi is parallel to a possib edge; hence the pla perpendicular to B co tains two lines parall to possible edges, ar is therefore parallel a possible face.

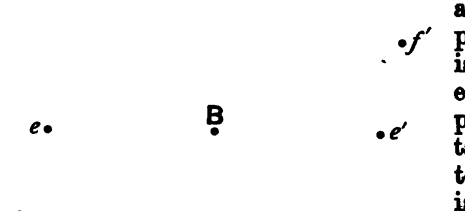

Fig. 18*.

(β). $n = 4$. Let OB the axis of symmetr and let Oe_1 be parallel to any possible edge. Let rotations abo OB through $\frac{\pi}{2}$ bring Oe_1, successively into the positions Oe Oe_3, Oe_4 (Fig. 19); then Oe_2, Oe_3, Oe_4 are by the symmetry al parallel to possible edges. Hence Oe_1e_3, Oe_2e_4 are parallel possible faces, and therefore their intersection OB is parall to a possible edge.

Again Oe_1e_4, Oe_2e_3 are parallel to possible planes and me in a line perpendicular OB which is parallel to possible edge; and the sam holds for Oe_1e_2, Oe_3e_4.

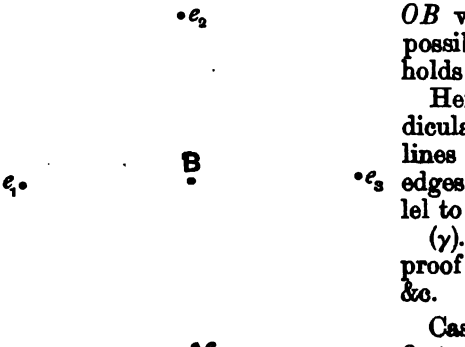

Fig. 19.

Hence the plane perpe dicular to OB contains tw lines parallel to possib edges and is therefore para lel to a possible face.

(γ). An exactly simil proof holds for $n = 6, 8, 1$ &c.

Case (2). n-al axes of t first sort; n odd.

(α). $n = 5$. Let Oe_1 parallel to any possible edg and let rotations about OB through $\frac{2\pi}{5}$ bring Oe_1 successivel to Oe_2, Oe_3, Oe_4, Oe_5: these lines are also parallel to possibl edges (Fig. 20).

* O is below B, so that OB is perpendicular to the plane of the paper.

Then the planes Oe_1e_5, Oe_3e_4 are parallel to possible faces, and therefore meet in a line Ol_2 parallel to a possible edge; similarly Oe_2e_3, Oe_4e_5 meet in a line Ol_1, parallel to a possible edge; hence Oe_1l_1, Oe_2l_2 are parallel to possible faces, and their intersection OB is parallel to a possible edge.

Again the planes Oe_1e_3, Oe_4e_5; Oe_2e_5, Oe_3e_4 are parallel to possible faces; the first two meet in a line perpendicular to OB, and so do the last two.

Hence the plane perpendicular to OB is parallel to two possible edges and is therefore parallel to a possible face.

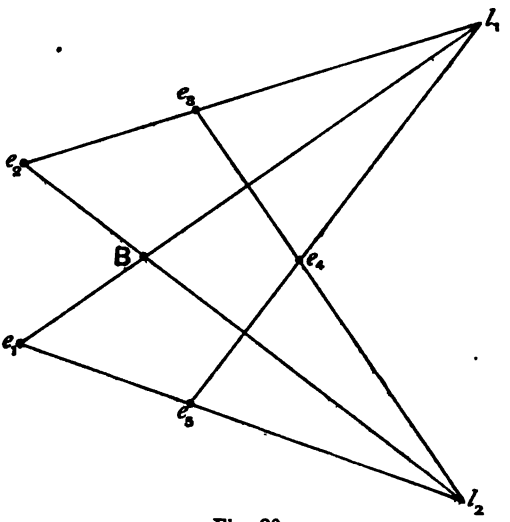

Fig. 20.

(β). An exactly similar proof holds for $n = 7, 9, 11,$ &c.

(γ). $n = 8$. In this case the theorem cannot be proved by aid of the law of rational indices alone; a proof will be given in Part II* with the help of the structure theory. The reason of the failure will be seen from the following discussion.

Let OX be parallel to any possible edge, and let rotations through $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ about the axis of symmetry bring OX to OY and OZ; these lines are also parallel to possible edges. Take OX, OY, OZ as crystallographic axes. Let the Cartesian coordinates of any point referred to these lines as axes of reference be x, y, z; and let the equation of any plane parallel to a crystal face be $Hx + Ky + Lz = 0$. Then by the symmetry $Kx + Ly + Hz = 0$ and $Lx + Hy + Kz = 0$ must also be parallel to crystal faces. The intercepts of these three faces on the crystallographic axes are inversely proportional to

$$1, \frac{K}{H}, \frac{L}{H}; \quad 1, \frac{L}{K}, \frac{H}{K}; \quad 1, \frac{H}{L}, \frac{K}{L}, \text{ respectively.}$$

* Pp. 144, 256.

Therefore by the law of rational indices, since 1 : 1 : 1 are rational ratios,

$$\frac{K}{H} : \frac{L}{K} : \frac{H}{L} \text{ and } \frac{L}{H} : \frac{H}{K} : \frac{K}{L}$$

must be also rational, i. e.

$$\frac{K^2}{LH} : 1 : \frac{HK}{L^2} \text{ and } \frac{LK}{H^2} : 1 : \frac{K^2}{LH}$$

must be rational ratios.

Hence $$\frac{H^2}{KL}, \quad \frac{K^2}{LH}, \quad \frac{L^2}{HK}$$

are rational, and therefore $H^3 : K^3 : L^3$ are rational ratios.

But to show that the plane $x+y+z=0$ (perpendicular to the axis of symmetry) is parallel to a possible face, we must know that $H : K : L$ are rational ratios*, which we cannot prove.

Note that if we know that the plane perpendicular to the symmetry-axis is parallel to a possible face, we may take its symbol as (1 1 1); and then the indices of the symmetry-axis are 1, 1, 1 which are rational, and hence the symmetry-axis is parallel to a possible edge.

Conversely, if the symmetry-axis OB is parallel to a possible edge we may take its symbol as [1 1 1].

Then the axial ratios are 1 : 1 : 1, for the direction-ratios of OB referred to OX, OY, OZ are evidently in the ratios 1 : 1 : 1 (p. 12).

Therefore the plane perpendicular to OB has the rational indices 1, 1, 1 and is parallel to a possible face.

There are some special cases of interest; for instance, we can prove the theorem if we know that there is a plane parallel to a crystal face, whose equation $Hx+Ky+Lz=0$ is such that $H^p : K^p$ is rational (p being an integer not divisible by 3).

For if $p = 3m+1$ (m integral), $\frac{H^{3m+1}}{K^{3m+1}}$ is rational and also $\frac{H^{3m}}{K^{3m}}$ is rational (for $\frac{H^3}{K^3}$ is rational). $\therefore \frac{H^{3m+1}}{K^{3m+1}} + \frac{H^{3m}}{K^{3m}} = \frac{H}{K}$ is rational.

* The intercepts made by a plane parallel to $x+y+z=0$ on the axes are in the ratio 1 : 1 : 1; those made by a plane parallel to $Hx+Ky+Lz=0$ are in the ratios, $\frac{1}{H} : \frac{1}{K} : \frac{1}{L}$; $\therefore$ by the law of rational indices, if both planes are parallel to possible faces, $H : K : L$ are rational ratios.

Similarly if $p = 3m-1$, $\frac{H^{3m}}{K^{3m}} + \frac{H^{3m-1}}{K^{3m-1}} = \frac{H}{K}$ is rational.

Now because $Hx+Ky+Lz = 0$ is parallel to a possible face, $\therefore$ by the law of rational indices $Hx+Hy+Lz = 0$ is also parallel to a possible face, since $H:K$ is rational. And by the symmetry $Hx+Ly+Hz = 0$, $Lx+Hy+Hz = 0$ are also parallel to possible faces.

Now these three planes meet in pairs in the lines Og_3, Og_2, Og_1, such that Og_1X, Og_2Y, Og_3Z are parallel to possible faces and are concurrent in the symmetry-axis, which is therefore parallel to a possible edge. By what we have proved above the perpendicular plane is parallel to a possible face.

Again we may prove the theorem if it is known that there are two lines through O which are parallel to possible edges and *equally inclined to the symmetry-axis**.

For let Oe_1, Of_1 be these lines and let them be brought by rotations through $\frac{2\pi}{3}$ about the symmetry-axis successively to the positions Oe_2, Of_2; Oe_3, Of_3 (Fig. 21). Then the plane passing through O and the intersections of the planes Oe_1f_2, Of_1e_3 and Oe_2f_2, Of_3e_3, and the plane through O and the intersections of the planes Of_2e_1, Oe_2f_3 and Of_1e_1, Oe_3f_3 are parallel to possible faces and meet in the symmetry-axis. This axis is therefore parallel to a possible edge.

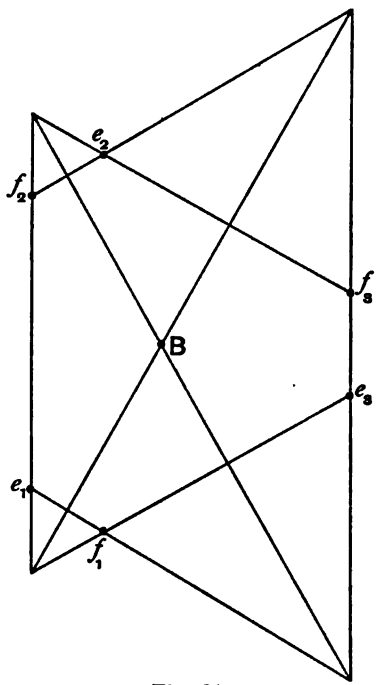

Fig. 21.

Case (3). n-al axes of the second sort; n even.

(a). In the case of $n = 2$ the theorem has no meaning,

* One of the lines can be brought into coincidence with the other by a rotation about the symmetry-axis through a certain angle. We assume that this angle is $\neq 0$, $\frac{2\pi}{3}$, or $\frac{4\pi}{3}$.

for a 2-al axis of the second sort is merely a centre of symmetry.

(β). $n>2$. Since an n-al axis of the second sort is an $\frac{n}{2}$-al axis of the first sort if n is even (§ 15, p. 36), every case except that of $n=6$ falls under case (1) (α), (β), or (γ), or under case (2) (α) or (β).

Now a 6-al axis of the second sort is equivalent to a 3-al rotation-axis combined with a centre of symmetry (p. 36); this centre of symmetry can make no difference to the results discussed under case (2) (γ); hence the theorem can no more be proved for a 6-al axis of the second sort than for a 3-al axis of the first sort.

Case (4). n-al axes of the second sort, n odd.

In this case the n-al axis of the second sort is equivalent to an n-al axis of the first sort *combined with a perpendicular symmetry-plane* (§ 15, p. 35); hence the theorem follows at once from (§ 17, p. 37).

It may be noticed that though we can find no proof of the above theorem in the case of a 3-al axis of the first sort or a 6-al axis of the second sort; yet in crystals of all substances hitherto investigated which have such an axis, either edges have been found which the most accurate measurements we can make cannot prove to be inclined to the axis of symmetry, or faces which cannot be shown not to be perpendicular to that axis.

Corollary 1. There are an indefinite number of possible edges perpendicular to any symmetry-axis b. For any possible face meets a plane perpendicular to b in a line parallel to a possible edge, since this plane is itself parallel to a possible face.

Corollary 2. If c be such a line through the centre O of the crystal perpendicular to an n-al axis b and parallel to a possible edge, and if c' be the position into which c would be brought by a rotation through $\frac{2\pi}{n}$ about b; then there is a possible face making equal intercepts on c and c'. ($n>2$).

For since b and c are parallel to possible edges, the plane through b and c is parallel to a possible face.

Let any other plane through O parallel to a possible face meet this plane in the line d which must be parallel to a possible edge.

First let b be a rotation-axis; and let d' be the position

into which d would be brought by a rotation through $\frac{2\pi}{n}$ about b. Then since d is parallel to a possible edge, by the symmetry of the crystal d' must also be parallel to a possible edge.

Therefore there must be a possible face parallel to the lines d and d', and this face makes equal intercepts on c and c'.

Next let b be an axis of the second sort; and let d' be the position into which d would be brought by a rotatory-reflexion through $\frac{2\pi}{n}$ about b. Then, as before, there is a possible face parallel to the lines d and d', and this face makes equal intercepts on c produced and on c'. Let the indices of this face referred to c, c', and b as crystallographic axes be $h\,k\,l$; then there is a possible face $(\bar{h}\,k\,l)$ making equal intercepts on c and c'.

§ 19. *The only possible symmetry-axes of a crystal are* 2-al, 3-al, 4-al, *and* 6-al *axes.*

Let b be an n-al symmetry-axis of a crystal (of the first or second sort), let c be a line through O the centre of the crystal perpendicular to b and parallel to a possible edge, and let c', c'' (Fig. 22 *) be the positions into which c would be brought by rotations through $\frac{2\pi}{n}$, $\frac{4\pi}{n}$ about b. Then by the symmetry of the crystal c' and c'' are also parallel to possible edges. Now there is a possible face making equal intercepts on c and c' (§ 18, Cor. 2); take this as parametral face, and take c, c', b as crystallographic axes (this is lawful since these lines are parallel to possible edges).

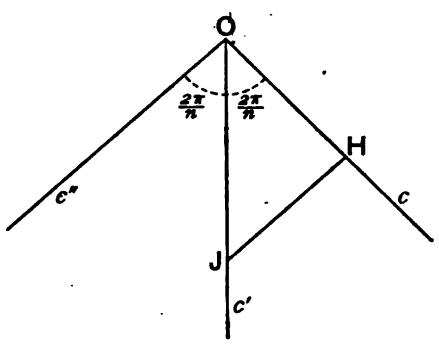

Fig. 22.

Since b and c'' are parallel to possible edges, there is a possible face parallel to these two lines; let it meet c and c' in H and J. Then the indices of this face are proportional to $\frac{1}{OH}$, $\frac{1}{OJ}$, 0; that is to $2\cos\frac{2\pi}{n}$, 1, 0.

* b is perpendicular to the plane of the paper in this figure.

Hence $2\cos\frac{2\pi}{n}$ is rational.

Now

$$\left(2\cos\frac{2\pi}{n}\right)^n - a_1\left(2\cos\frac{2\pi}{n}\right)^{n-2} + a_2\left(2\cos\frac{2\pi}{n}\right)^{n-4}$$
$$- a_3\left(2\cos\frac{2\pi}{n}\right)^{n-6} + \ldots = 2\cos n\left(\frac{2\pi}{n}\right) =$$

where $a_r = {}^{n-r}C_r + {}^{n-r-1}C_{r-1}$ and is integral *.

Therefore $2\cos\frac{2\pi}{n}$ cannot be of the form $\frac{p}{q}$ where p and are integers prime to one another; for in that case we shou have

$$\frac{p^n}{q^n} - a_1\frac{p^{n-2}}{q^{n-2}} + a_2\frac{p^{n-4}}{q^{n-4}} - \ldots = 2$$

or

$$\frac{p^n}{q} = 2q^{n-1} + a_1 p^{n-2} q - a_2 p^{n-4} q^3 + \ldots,$$

a fraction equal to an integer, which is impossible.

But $2\cos\frac{2\pi}{n}$ is rational, and we have just shown that is not fractional; therefore $2\cos\frac{2\pi}{n}$ is integral. Also $2\cos\frac{2}{}$ is numerically not greater than 2; therefore $\cos\frac{2\pi}{n} = 0, \pm$ or ± 1, whence we deduce $n = 2, 3, 4$, or 6.

* See Hobson's "Plane Trigonometry," p. 103.

CHAPTER IV

THE THEORY OF GROUPS.

§ 1. A series of operations is said to form a *group* when (1) the product of any two of the operations, or the square of any one, is equivalent to some member of the series *, and (2) the series always contains the operation A^{-1} if it contains the operation A.

Every group contains the identical operation, for $A \,.\, A^{-1} = 1$.

A group is said to be *finite* or *infinite* according as the number of operations it contains is finite or infinite.

The product of any number of operations of a group is itself a member of the group.

For if $A.B.C.D \ldots K$ is the product of any number of operations A, B, C, ... of a group, not necessarily all distinct, then this product $= M.C.D \ldots K$ (where M is a member of the group); $= N.D \ldots K$ (where N is a member of the group); $= H.K$ (where H is a member of the group, by repetition of the process); $= J$ (where J is an operation of the group).

In particular any integral power of any operation of a group is a member of that group.

As an example of a group of operations we may take a rotation A through π about an axis a, a reflexion S in a plane σ perpendicular to a, and an inversion I about the intersection of a and σ. These movements form a group, for

$$A.S = S.A = I; \quad I.S = S.I = A; \quad I.A = A.I = S;$$
$$A^2 = S^2 = I^2 = 1.$$

§ 2. A group is said to be of the second or of the first sort according as it does or does not contain operations of the second sort.

Let $L_1, \quad L_2, \quad L_3, \quad L_4, \quad \ldots\ldots\ldots$ (i)

be the distinct operations of the first sort of a group of the second sort, and let L', M be any two operations of the second sort contained in the group. Then L'^2 and M^2 are of the first sort †, and are therefore included in (i).

Now $M.L'$ is an operation of the first sort † of the group; suppose $M.L' = L_u$.

Then $M.L'.L_u^{-1} = 1$; hence $L'.L_u^{-1} = M^{-1}$, and therefore

* i. e. some operation of the series. † See p. 30.

$M = L' . L_u^{-1} . M^2 = L' . L_p$; where $L_p = L_u^{-1} . M^2$ is included in (i).

Again $L' . L_p \neq L' . L_j$, for $L_p \neq L_j$; hence we conclude that

$L' . L_1, \quad L' . L_2, \quad L' . L_3, \quad L' . L_4, \ . \ . \ . \ . \ . \ . \ . \ .$ (ii)

are all distinct and include all operations of the second sort of the group, L' being *any* operation of the second sort contained in the group.

Similarly $L_1 . L', \quad L_2 . L', \quad L_3 . L', \quad L_4 . L', \ . \ . \ .$ (iii)

are all distinct and include all operations of the second sort contained in the group. The series (iii) is the same as (ii), though the order of the operations in the two series is in general different.

It follows that:—

Every finite group of the second sort contains as many operations of the first sort as of the second sort.

§ 3. It sometimes happens that a certain portion of the operations of a group form a group when taken alone; the portion is then said to form a *subgroup*.

If a group contains operations of the first and of the second sort; the operations of the first sort form a subgroup.

For the product of two operations of the first sort is an operation of the first sort (p. 30).

§ 4. Let $L_1, \quad L_2, \quad L_3, \quad L_4, \ . \ . \ . \ . \ . \ . \ . \ .$ (i)

be the distinct operations of any subgroup G_1, of a group G.

Let M be any operation of G not contained in (i).

Then $M . L_1, \quad M . L_2, \quad M . L_3, \quad M . L_4, \ . \ . \ .$ (ii)

are operations distinct from each other and from the operations (i). For $M . L_u$ cannot equal* $M . L_v$ for then L_u would equal L_v, and $M . L_u$ cannot equal L_p for then M would equal $L_p . L_u^{-1}$, that is, would equal some operation of the series (i).

Let M' be an operation of G not included in (i) and (ii) (if such an operation exists).

Then $M' . L_1, \quad M' . L_2, \quad M' . L_3, \quad M' . L_4, \ . \ . \ .$ (iii)

are operations of G distinct from each other and from the operations (i) and (ii). For as before $M' . L_u$ cannot equal $M' . L_v$ or L_p, and it cannot equal $M . L_w$ for then M' would equal $M . L_w . L_u^{-1}$, that is, some operation of the series (ii).

If there is some operation of G not contained in (i), (ii), (iii), we proceed as before. If G is an infinite group it may or may not be possible to form an infinite number of rows similar to

* Two operations are said to be 'equal' or 'equivalent' when they have identical effects on *any* figure.

(i), (ii), &c. If, however, G (and therefore G_1) contains only a finite number of operations, we must at length reach a row (p) such that it is impossible to find an operation of G different from all those in (i) (ii) ... (p).

Now all the operations in these p rows are distinct and belong to G. Hence the number of operations of G is p times that of G_1, and therefore

The number of operations of a finite group is a multiple of the number of operations of any subgroup.

§ 5. Let a be any point, line, or plane, and let a be brought to coincidence with $a_1, a_2, a_3, \ldots$, respectively, by the operations $L_1, L_2, L_3, \ldots$ of any group. Then $a, a_1, a_2, a_3, \ldots$ are said to be a system of *equivalent* * points, lines, or planes. Any operation L_h of the group is a symmetry-operation of this system; for, if a_p is any member of the system and $L_p . L_h = L_e$, L_h brings a_p to coincide with a_e †.

Hence *the operations of any group are symmetry-operations of an infinite number of different figures.*

It should be noticed that a_p cannot be brought to coincide with a_e by two distinct operations L_h and L_k; for then $L_h . L_k^{-1}$ would leave a_p unmoved, which is not in general the case unless $L_h . L_k^{-1} = 1$, i.e. $L_h = L_k$. We deduce immediately (taking a_p as a) that *the number of points, lines, or planes in an equivalent system of a finite group is in general equal to the number of operations in that group.*

§ 6. It follows from p. 33 that *the symmetry-operations of any figure form a group* ‡; for they satisfy the conditions of p. 45.

Let L be any symmetry-operation of a figure U, and let any symmetry-element b of U be brought into the position b' by L. Then, since U, b are brought into coincidence with U, b' respectively by L, therefore b' is related to U in the same way that b is. Therefore b' is a symmetry-element of U of the same kind as b. Hence *the symmetry-operations of any figure are symmetry-operations of the system formed by all the symmetry-elements of the same kind possessed by the figure.*

For example any symmetry-operation of U brings any n-al rotation-axis of U into coincidence with itself or some other

* German—gleichwerthig.

† L_p^{-1} brings a_p to coincide with a, and L_e brings a to coincide with a_e; therefore $L_p^{-1} . L_e$ brings a_p to coincide with a_e.

‡ This is the converse of the first theorem of the previous section.

n-al rotation-axis of U, any symmetry-plane of U into coincidence with itself or some other symmetry-plane of U, &c.

This does not imply that any given n-al rotation-axis of U can be brought into coincidence with any other by a symmetry-operation of U, &c. A system of elements of the same kind such that any one can be brought into coincidence with any other by some symmetry-operation of U is called an 'equivalent system of symmetry-elements' of that kind.

By 'symmetry-elements of a group' we shall mean the axes, planes, &c., about which the operations of the group act; that is, the symmetry-elements of all figures which are brought to self-coincidence by every operation of the group. Similarly we talk of 'rotation-axes,' 'symmetry-planes,' &c. of a group. We say also that a group 'contains,' 'has,' or 'possesses' symmetry-elements.

§ 7. It is evident that a 'crystal' (see p. 36) can only be brought to self-coincidence in a finite number of ways; and therefore *the symmetry-operations of a crystal form a finite group.*

Such a group can only possess rotations about axes passing through a fixed point O (the 'centre' of the group), reflexions in planes through O, rotatory-reflexions whose axes meet the corresponding planes at O, and the inversion about O. All the operations of the group leave the point O unmoved.

Finite groups whose operations all leave one point unmoved are called *point-groups*. By 'group' we shall mean 'point-group' for the remainder of Part I.

§ 8. We proceed to prove certain properties of a system of equivalent points of a group (i.e. a point-group). Similar theorems will be seen to hold good for a system of equivalent lines or planes.

The number of points in the system is *in general* equal to the number of operations in the group (p. 47). If, however, one of the points has some special position, this is not necessarily true.

If one of the points of the system lies in a symmetry-element, every other point of the system lies in a symmetry-element of the same kind; for every operation of the group brings the system of symmetry-elements of the same kind to self-coincidence (p. 47).

Conversely if one of the points P does *not* lie in a symmetry-element of a given kind no other point P' of the system can do so; for if P' lay in a symmetry-element of the given kind so would P; and this is contrary to hypothesis.

For example if one of the points of the system lies in an n-al rotation-axis so do all the rest; if one of the points is not in an n-al rotation-axis no other point can lie on such an axis.

§ 9. *If one of the points of an equivalent system lies in a symmetry-plane and in no other symmetry-element, the number of points in the system is half that of the operations of the group.*

For if one of the points lies in a symmetry-plane and in no other element, it coincides with its reflexion in that plane, but with no other point of the system. Moreover any other equivalent point also lies in a symmetry-plane and in no other element, and coincides with its reflexion in that plane, but with no other point of the system.

§ 10. If A is a rotation through $\frac{2\pi}{n}$ about an n-al rotation-axis a, and P is a point on that axis, the n points to which P is brought by the operations $1, A, A^2, \ldots, A^{n-1}$ coincide. Also each of the points equivalent to P lies in an n-al rotation-axis equivalent to a; hence it follows that

If one of the points of an equivalent system lies in an n-al rotation axis, but in no other symmetry-element, the number of points in the system is $\frac{1}{n}$ the number of operations in the group.

Now suppose a lies in a symmetry-plane σ, and let S represent a reflexion in this plane; let P be a point on a. Then the $2n$ points to which P is brought by the operations $1, A, A^2, \ldots, A^{n-1}, S, S.A, S.A^2, \ldots, S.A^{n-1}$ coincide.

Also each of the equivalent points lies in an n-al rotation-axis equivalent to a and in a plane of symmetry equivalent to σ; hence we have

If one of the points of an equivalent system lies in an n-al axis of symmetry through which passes a plane of symmetry, the number of points in the system is $\frac{1}{2n}$ the number of operations in the group.

§ 11. A rotation-axis is said to be *two-sided* or *one-sided* * according as points equidistant from and on opposite sides of the centre are or are not equivalent.

* German—zweiseitig, einseitig.

A two-sided axis must evidently have a symmetry-plane or a $2m$-al axis perpendicular to it (where m is a positive integer).

The number of operations in a group which contains a equivalent n-al rotation-axes is in general $2na$ or na, according as the axes do or do not lie in symmetry-planes; where two-sided axes count for two separate axes in reckoning the number a.

If a point P lies in one of the a n-al axes, so do all points equivalent to P; for the axes are equivalent. Therefore the number of points equivalent to P is in this case a. But (p. 49) this number is $\frac{1}{2n}$ or $\frac{1}{n}$ the number of operations in the group, according as the a n-al axes do or do not lie in symmetry-planes. Therefore the number of operations is $2na$ or na according as the a equivalent axes do or do not lie in symmetry-planes.

§ 12. *If the operations of the first sort of a group consist of the identical operation together with rotations about V_2 2-al axes, V_3 3-al axes, V_4 4-al axes, ... (where a two-sided axis counts only as a single axis in reckoning V); then the number of operations contained in the group is*

$$1 + V_2 + 2V_3 + 3V_4 + \dots$$

or twice that number according as the group is of the first or second sort.

For any point P is brought to coincide with $(n-1)$ equivalent points by rotation about an n-al axis; hence if rotations are the only operations of the group there must be at least

$$V_2 + 2V_3 + 3V_4 + \dots$$

points equivalent to P, since it is impossibl... uld be brought into coincidence with the same ... ations through an angle less than 2π about two ... s both passing through the centre of the crystal C ... there cannot be more than $V_2 + 2V_3 + 3V_4 + \dots$ ... valent to P, for P is brought into coincidence with ... valent point by a rotation about *some* axis belon ... group. Hence if there are no operations of the sec... whole number of operations including the identica...

$$1 + V_2 + 2V_3 + 3V_4 + \dots .$$

If on the other hand the group contains ... of the second sort the whole number of operatio ... hat of the operations of the first sort (p. 46). N... just

proved that the number of operations of the first sort is $1 + V_2 + 2V_3 + 3V_4 + \ldots$, and therefore the whole number of operations is $2(1 + V_2 + 2V_3 + 3V_4 + \ldots)$.

§ 13. The points of an equivalent system are evidently all equidistant from the centre of the group, and therefore all lie on a sphere. Consequently we may form a stereographic projection of the system, and such a projection often forms a useful representation of the group.

The projection of the great circle in which the sphere is cut by any symmetry-plane is denoted by a continuous line; and the projection of the great circle parallel to the plane of projection (if not a symmetry-plane) by a broken line. The projections of points in which the sphere is met by a 2-al, 3-al, 4-al, or 6-al rotation-axis are marked by the symbols , , , , respectively; while the projections of points in which the sphere is met by a 4-al or 6-al axis of the second sort are marked by , , respectively.

CHAPTER V

FINITE GROUPS OF THE FIRST SORT.

§ 1. In this chapter we shall investigate those finite groups of the first sort—also called holoaxial groups—which contain only 2-al, 3-al, 4-al, and 6-al rotation-axes. No group of symmetry-operations of a crystal whose faces obey the law of rational indices can contain rotation-axes other than these.

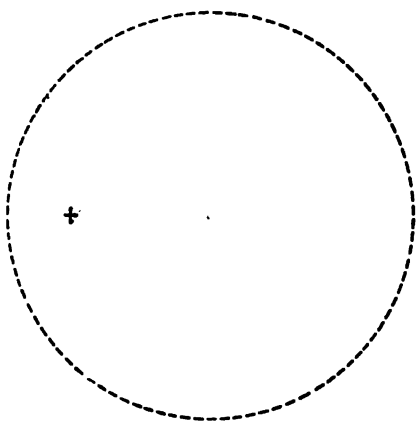

Fig. 23. C_1.

In each case we shall give a stereographic projection of a system of equivalent points (Figs. 23–31, and 33, 34).

The first group to be considered is that containing no operation of symmetry at all (Fig. 23). The number of points in an equivalent system is one. We denote this group by the symbol C_1; it contains only the identical operation.

§ 2. We have next the four groups with a single 2-al, 3-al, 4-al, or 6-al axis (Figs. 24–27; the axis is perpendicular to the plane of projection).

We denote these groups by the symbols C_2, C_3, C_4, C_6 respectively.

If A is a rotation through $\frac{2\pi}{n}$ about the axis of symmetry the groups contain the operations $1, A, \ldots, A^{n-1}$ where $n = 2$, 3, 4, or 6.

§ 3. Consider next a group containing *one* n-al axis a and a perpendicular 2-al axis b.

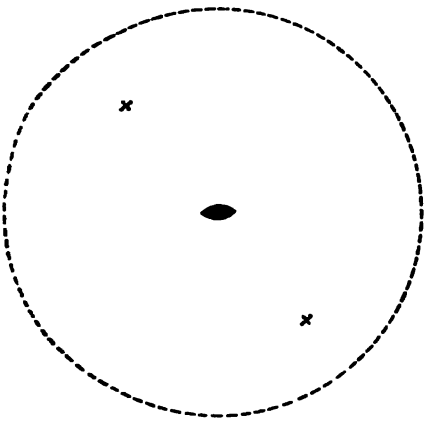

Fig. 24. C_2.

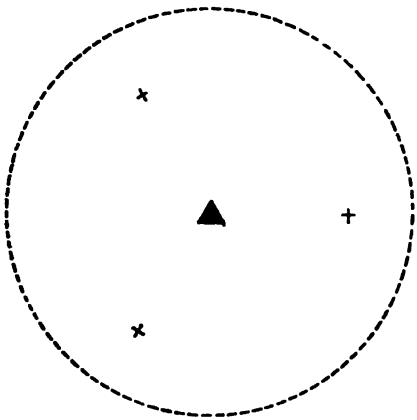

Fig. 25. C_3.

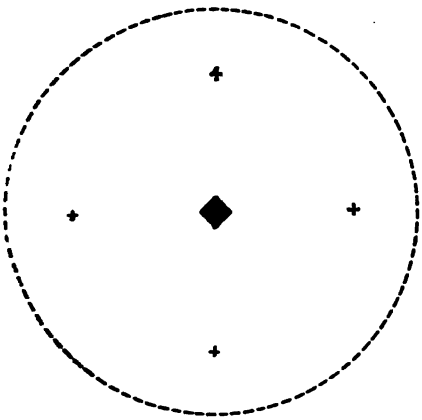

Fig. 26. C_4.

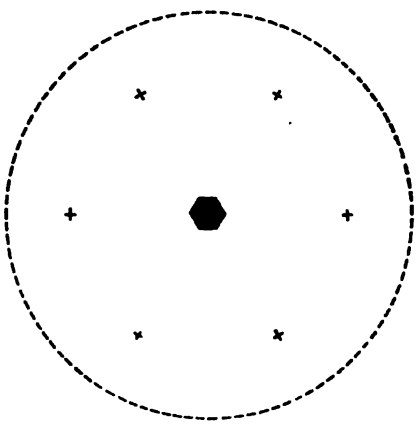

Fig. 27. C_6.

Since each operation of the group brings the system of symmetry-elements into self-coincidence, the positions $b', b'' \ldots$ into which b would be brought by successive rotations through $\frac{2\pi}{n}$ about a, $(n-1)$ in number if n is odd and $\left(\frac{n}{2}-1\right)$ in number if n is even, must be also occupied by 2-al axes of symmetry.

Now the resultant of the rotation through $\frac{2\pi}{n}$ about a followed by the rotation through π about b is readily seen by

Euler's construction * to be a rotation through π about a line c perpendicular to a and making an angle $\frac{\pi}{n}$ with b. If n is odd c coincides with one of the series $b, b', b'', \ldots$, but if n is even there must be a series of $\frac{n}{2}$ 2-al axes $c, c', c'', \ldots$ bisecting the angles between $b, b', b'', \ldots$. In either case there can be no other symmetry-axis; for if a is the *only* n-al axis there can be no axes except a itself and 2-al axes perpendicular to a; and there can be no such 2-al axes not belonging to the series $b, b', b'', \ldots$ †, or to the series $c, c', c'', \ldots$ †. For if there were such an axis let β be the angle between it and that axis of the series $b, b', b'', \ldots$ which makes the smallest angle with it; then since

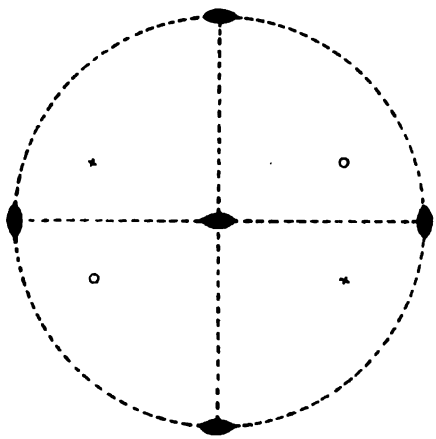

Fig. 28. $D_2 = Q$.

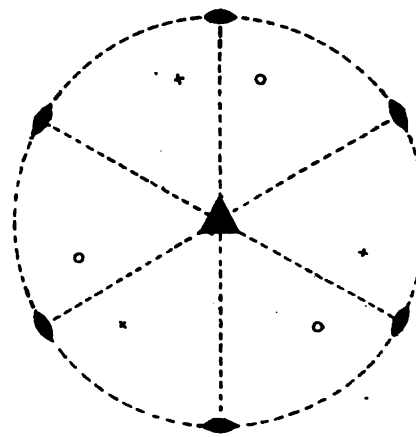

Fig. 29. D_3.

there are two 2-al axes perpendicular to a making an angle β with one another; therefore a is a $\frac{\pi}{\beta}$-al axis ‡ contrary to hypothesis (for since $\beta < \frac{\pi}{n}$, $\therefore \frac{\pi}{\beta} > n$).

It is readily seen that the rotations about a; $b, b', b'', \ldots$ $c, c', c'', \ldots$ † form a group; and that $b, b', b'', \ldots$ are equivalent axes, and so are $c, c', c'', \ldots$; but that none of the b's is equivalent to any one of the c's (if n is even).

The total number of 2-al axes is always n, whether n odd or even.

* See p. 24.

† These two series coincide if n is odd.

‡ This also follows from Euler's construction, see end of § 8, chap. iii, p. 1

If A be the rotation through $\frac{2\pi}{n}$ about a, and B the rotation through π about b, the operations of the group are evidently

$$1, \quad A, \quad A^2, \quad \ldots, \quad A^{n-1},$$
$$B, \quad A.B, \quad A^2.B, \quad \ldots, \quad A^{n-1}.B.$$

Taking $n = 2$, we have a group consisting of three mutually perpendicular 2-al axes (Fig. 28). This has been called the 'Quadratic' group *. We shall represent it by Q or D_2 †.

Taking $n = 3$, 4, and 6, we have three groups which we shall denote by D_3, D_4, and D_6, respectively (Figs. 29–31; in these figures the axis a is perpendicular to the plane of pro-

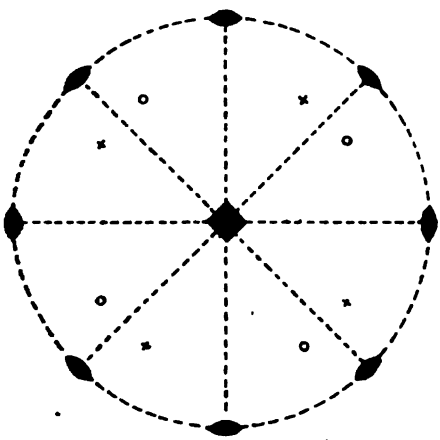

Fig. 30. D_4.

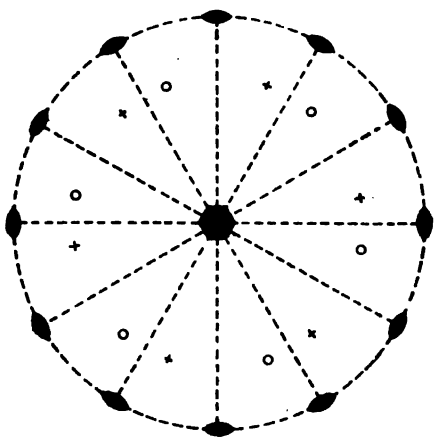

Fig. 31. D_6.

jection, and the lines in which planes through the vertex of projection and the 2-al axes cut the plane of projection are represented by broken lines).

§ 4. Consider now the most general case. Draw a sphere whose centre is O, the centre of the crystal. Let any n-al axis of the group meet the sphere in e, and let Od be any axis of the group which is not a 2-al axis perpendicular to Oe. Then the rotation about Od brings Oe into the position of some n-al

* Introduced by G. G. Morrice as the English equivalent of the German 'Vierergruppe' in his translation of F. Klein's "Lectures on the Icosahedron."

† D_2 points out the analogy with D_3, D_4, D_6; Q the fact that all its axes have similar properties.

axis equivalent to Oe (for the series of symmetry-elements is brought into self-coincidence by the rotation about Od). Hence the group contains a series of equivalent n-al axes. Let Oe and Oe' be two of these axes, such that no two equivalent n-al axes of the group make an angle with each other smaller than the acute angle eOe'. Then if the rotation through $\frac{2\pi}{n}$ about Oe' brings Oe into the position Oe'' (Fig. 32), Oe'' is an n-al axis equivalent to Oe and Oe'. Similarly the rotation through $\frac{2\pi}{n}$ about Oe'' brings Oe' into the position of another equivalent n-al axis Oe''', and proceeding in this way we see that the equivalent axes meet the sphere in points e, e', e'', ... forming a regular spherical polygon. This polygon must be closed, for otherwise there would be two equivalent axes of the system, making with each other an angle $< eOe'$. Let p be the number of sides of the polygon. The resultant of the rotation through $\frac{2\pi}{n}$ about Oe, followed by an equal rotation about Oe', is by Euler's construction a rotation about Of (where f is the pole of the small circle on which e, e', e'', ... lie) through an angle $-\frac{4\pi}{p}$. Now the area of the polygon $ee'e''$... is $[2n-p(n-2)]\frac{K}{4n}$, where K is the area of the whole surface of the sphere, and this must be positive.

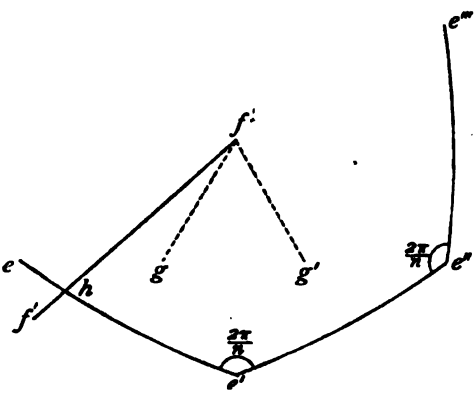

Fig. 32.

The only possible values of n are 3, 4, 6 (rejecting $n=$ which gives us the group Q of § 3); the only values of which make $2n-p(n-2)$ positive are

for $n=3$, $p=2$, 3, 4 or 5;
for $n=4$, $p=2$ or 3;
for $n=6$, $p=2$.

We reject $p=2$ which gives us the groups of § 3, and also $p=5$ which would make the line Of of Fig. 32 a 5-al axis and so inconsistent with the law of rational indices; we are left with

$$n=3,\ p=3 \text{ or } 4;$$
$$n=4,\ p=3.$$

In the first case ($n=3, p=3$) the polygon is an equilateral spherical triangle whose angles are $\frac{2\pi}{3}$, and whose sides are $\cos^{-1}(-\frac{1}{3})$ (this is easily deduced by solving the spherical triangle efe' all of whose angles are known); we must reject this case, for since the angle eOe' is acute, its cosine cannot be negative.

In the second case ($n=3$, $p=4$) the angle eOe' is seen by solving the triangle efe' to be $\cos^{-1}\frac{1}{3}$. This is the angle subtended at the centre of the sphere by a side of an inscribed cube. Hence if we produce eO, $e'O$, $e''O$, $e'''O$ to meet the sphere again in e^{iv}, e^{v}, e^{vi}, e^{vii}, the eight points $e, e', \ldots, e^{viii}$ lie at the vertices of an inscribed cube. The group can contain no other 3-al axis; for evidently a rotation through $\frac{2\pi}{3}$ about any line other than eO, $e'O$, $e''O$, or $e'''O$, does not bring this system of four equivalent 3-al axes to self-coincidence.

In the third case ($n=4, p=3$) $eOe'=\frac{\pi}{2}$, and hence eO, $e'O$, $e''O$, are mutually perpendicular, and if produced they cut the sphere at e''', e^{iv}, e^{v}. As before the group can contain no other 4-al axis.

§ 5*. We can treat the problem in a more general way without assuming that $n=2$, 3, 4, or 6 as follows.

The rotation about Oe will bring the polygon $ee'e''\ldots$ of Fig. 32 into a new position, and, by repeating the process of rotation about various equivalent axes, we can completely cover the surface of the sphere with polygons whose angular points lie on axes equivalent to Oe. Now these polygons do not overlap; for (1) if starting by rotating the polygon about Oe we could arrive at a polygon, one of whose angular points g lay inside $ee'e''e'''\ldots$, but did not coincide with f; by starting from a rotation about Oe' and pursuing a precisely similar course we could arrive at a polygon, one of whose angular points g' is situated relatively to e' as g is to e (Fig. 32). Then since $fe=fe'$, $fg=fg'$, and the angle $efe'=gfg'$; but

* §§ 5 and 6 may be omitted without affecting the subsequent argument.

$fg < fe < \frac{\pi}{2}$; therefore $gg' < ee'$; contrary to the hypothesis that eOe' is the smallest angle between equivalent n-al axes of the group.

(2) If f is an angular point of one of the polygons and f' an angular point next to it; then f' cannot coincide with one of the points $e, e', e'' \ldots$, for otherwise (since $e'fe$ is then equilateral) the smallest angle of rotation about Oe, i.e. the angle $2(fee')$, would be $> \frac{2\pi}{3}$; also f' cannot lie inside the polygon $ee'e'' \ldots$ as we have just proved, nor evidently on an edge of it. Suppose therefore that f' lies outside the polygon, and let ff' cut an edge (say ee') in h. Of eh and $e'h$ not both can be the greater; suppose $eh \geqslant e'h$ and similarly suppose $f'h \geqslant fh$. Then $ef' < eh + f'h < \frac{1}{2}(ee' + ff') < ee'$ (since $ee' = ff'$), i.e. Of' makes a smaller angle with Oe than Oe' does, contrary to hypothesis.

(3) It is impossible for one of the polygons to overlap $ee'e'' \ldots$ so that one of its angular points lies at e and another adjacent angular point lies at e'' (say), for evidently $ee'' > e$

The sphere then is completely covered with non-overlapping polygons whose vertices are the points where axes equivalent to Oe meet the sphere. Let S be the number of such points, E the number of polygon edges and F the number of the polygons. Now the area of one of the polygons is $[2n - p(n-2)]\frac{K}{4n}$*$= \frac{\pi}{2}$. Hence

$$F = \frac{4n}{2(p+n) - pn}.$$

As F is positive, $2(p+n) > pn$ or $\frac{1}{p} + \frac{1}{n} > \frac{1}{2}$; but (rejecting n or $p = 2$ which leads to groups considered in § 3) $\frac{1}{n}$ is $\geqslant \frac{1}{3}$; and therefore $\frac{1}{p}$ must be $< \frac{1}{6}$; similarly $\frac{1}{n}$ must be $< \frac{1}{6}$. Hence n and p are both > 6.

It is found at once on trial that the only values of p and n which make F a positive integer are

$$n = 3, \quad p = 3, 4 \text{ or } 5;$$
$$n = 4 \text{ or } 5, \quad p = 3.$$

We have also $pF = nS = 2E$ for each quantity $= \sum_i \gamma_i$ the to

* See § 4, p. 56.

number of polygon angles (such as $e\,e'\,e''$), and therefore

$$S = \frac{4p}{2(p+n)-pn}, \quad E = \frac{2pn}{2(p+n)-pn};$$

from these and $F = \dfrac{4n}{2(p+n)-pn}$ we at once deduce the relation $S + F = E + 2$.

We have then the following table:—

n	p	S	E	F	*The axes meet the sphere at the vertices of a regular*
3	3	4	6	4	Tetrahedron
3	4	8	12	6	Hexahedron or Cube
4	3	6	12	8	Octahedron
3	5	20	30	12	Dodecahedron
5	3	12	30	20	Icosahedron

We see at once from the values of n, p, s, E and F given in the table that the centres of the faces of a regular tetrahedron are the vertices of another regular tetrahedron; that the centres of the faces of a regular hexahedron are the vertices of a regular octahedron and vice versa; and that the same reciprocal relation is true of the dodecahedron and icosahedron.

§ 6. We proved in § 4 that the resultant of a rotation through $\frac{2\pi}{n}$ about Oe followed by a similar rotation about Oe' is a rotation through $\frac{-4\pi}{p}$ about Of. If p is odd this means that Of is a p-al axis; if p is even Of is either a $\frac{p}{2}$-al axis or a p-al axis. If therefore a group contains six 5-al axes passing through the vertices of a regular icosahedron it contains also ten 3-al axes passing through the vertices of a regular dodecahedron, and vice versa. The group also contains fifteen 2-al axes bisecting the edges of the icosahedron and dodecahedron, for the resultant of a rotation through $\frac{2\pi}{5}$ about a 5-al axis followed by a rotation through $\frac{2\pi}{3}$ in the same direction about a neighbouring 3-al axis is readily seen by Euler's construction to be a rotation through π about such a 2-al axis.

The rotations above-mentioned can easily be seen to form a group; it must contain sixty operations (p. 50). The group

contains 5-al axes, and therefore cannot be a group of symmetry-operations of a crystal obeying the law of rational indices.

§ 7. We now turn to the second case of § 4 ($n = 3, p = 4$). In this case there are four 3-al axes lying along the diagonals of a cube, and the axes perpendicular to the faces of the cube are 2-al axes * (for a rotation through $\frac{4\pi}{p}$ about Of in Fig. 32 was proved in § 4 to be an operation of the group).

The rotations about these four 3-al axes and three 2-al axes are readily seen (by Euler's construction or otherwise) to form a group.

The system of equivalent points is shown in Fig. 33; we represent the group by the symbol T.

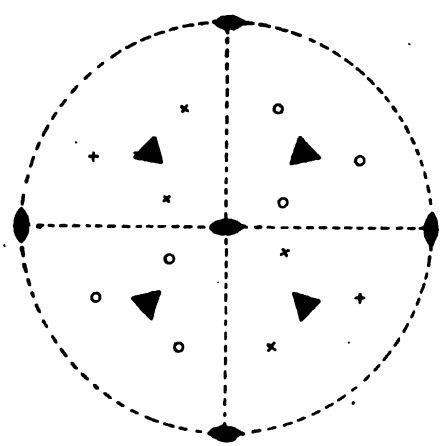

Fig. 33. T.

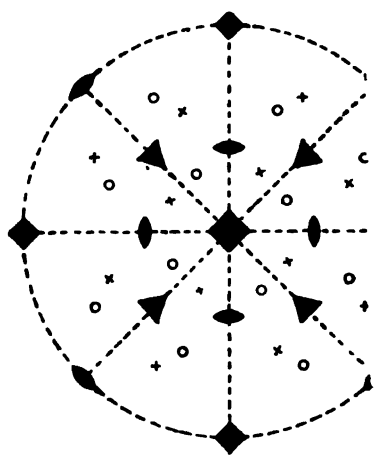

Fig. 34. O.

§ 8. Consider now the third case of § 4 ($n = 4, p = 3$). We have three 4-al axes lying along the diagonals of an octahedron, or, what is the same thing, perpendicular to the faces of a cube. These necessitate the existence of four 3-al axes perpendicular to the faces of the octahedron, or lying along the diagonals of the cube (for a rotation through $\left(2\pi - \frac{4\pi}{p}\right)$ about Of was proved in § 4 to be an operation of the group).

A rotation through $\frac{2\pi}{p}$ about Of (Fig. 32) followed by a

* Or may be 4-al; we shall return to this case later.

rotation in the same direction through $\frac{2\pi}{n}$ about Oe is readily seen by Euler's construction to be equivalent to a rotation through π about an axis bisecting ee'; hence in the case under consideration there must be six 2-al axes bisecting the angles between the 4-al axes (and also the angles between neighbouring 3-al axes).

The 4-al, 3-al, 2-al rotations above-mentioned are readily seen to form a group, which we shall represent by the symbol O.

If U, V, W denote rotations through π about the three 4-al axes, A a rotation through $\frac{2\pi}{3}$ about a 3-al axis, and R a rotation through π about a 2-al axis, the operations of the group may be written

$$\begin{array}{llll} \left\{\begin{array}{l}1,\\ R,\end{array}\right. & U, & V, & W;\\ & U.R, & V.R, & W.R;\\ A, & U.A, & V.A, & W.A;\\ A.R, & U.A.R, & V.A.R, & W.A.R;\\ A^2, & U.A^2, & V.A^2, & W.A^2. \quad . \quad . \quad . \quad \text{I}\\ A^2.R, & U.A^2.R, & V.A^2.R, & W.A^2.R. \quad . \quad . \quad \text{II} \end{array}$$

The first lines contain the operations of T which is a subgroup of O.

The projection of a system of equivalent points is given in Fig. 34.

§ 9. There are no other possible finite groups of the first sort. For in §§ 4 and 8 we proved that any group other than those of §§ 1, 2, 3, and 6 must contain four 3-al axes lying along the diagonals of a cube. Now any operation of a group containing these 3-al axes must bring these axes into self-coincidence and must therefore bring the cube of which they are the diagonals into self-coincidence; but evidently the only operations which can do this are the operations of the groups T and O.

NOTE.—The results of § 4 may be obtained without assuming the formula for the area of a spherical polygon. Since the angle efe' (Fig. 32) must have one of the values $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}$, we have only to see which of these values make $\cos ee'$ positive and >1, when (1) $fee' = \frac{\pi}{3}$, (2) $fee' = \frac{\pi}{4}$; given $\sin^2 fee' . \cos ee' = \cos^2 fee' + \cos efe'$.

CHAPTER VI

FINITE GROUPS OF THE SECOND SORT.

§ 1. Since the operations of the first sort of a group G form a subgroup G_1; and since the operations of the second sort of G can be found by multiplying the operations of G_1 by *any* operation of the second sort (H) belonging to G*, and since the elements of symmetry are brought into self-coincidence by any operation of G; therefore we can find the possible groups of the second sort by combining the operations of any group G_1 of chapter v with an operation of the second sort which brings its system of rotation-axes into self-coincidence. It will be readily verified in each case that the operations so obtained really form a group †.

Since H^2 is an operation of the first sort contained in G, therefore H^2 is an operation of G_1.

It follows from p. 35 that if a group contains a 3-al axis of the second sort it contains a plane of symmetry; and that if it contains a 6-al axis of the second sort it contains a centre of symmetry. Hence the only operations of the second sort to be combined with the operations of the groups of chapter v, in order to get all the groups of the second sort are a reflexion in some plane, an inversion about the centre O, and a rotatory-reflexion of angle $\frac{\pi}{4}$ about some line through O. A rotation through π about this line is an operation of the group; the line must therefore coincide with a 2-al, 4-al, or 6-al rotation-axis. If, however, it coincides with a 4-al rotation-axis, the group must contain a reflexion; for a rotatory-reflexion through $\frac{\pi}{4}$ combined with a rotation through $\frac{\pi}{4}$ in the opposite direction about the same line is equivalent to a reflexion in a plane perpendicular to that line.

* P. 46.

† A proof (other than mere verification in each case) will be given for a more general case later; see p. 168.

If it coincides with a 6-al rotation-axis that axis is readily seen by similar reasoning to be a 12-al axis of the second sort, which is impossible in a group applicable to crystallography.

§ 2. We have first of all a group C_s containing the two

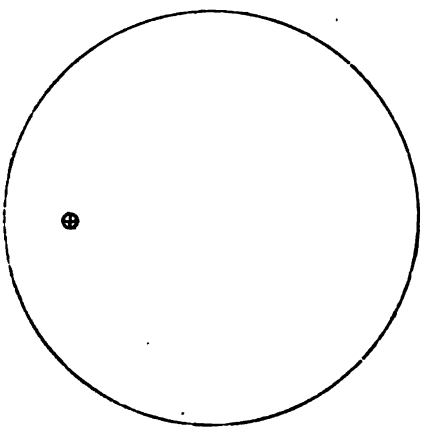

Fig. 35. C_s.

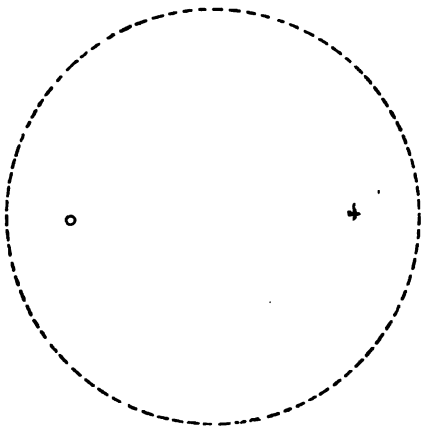

Fig. 36. C_i.

operations 1, S; i.e. the identical operation and a reflexion in one plane of symmetry. (Fig. 35).

Next we have a group C_i containing the two operations 1, I; i.e. the identical operation and inversion about a centre of symmetry. (Fig. 36). This group is that containing a 2-al axis of the second sort.

Then we have a group C_4' containing the operations 1, A', A'^2, A'^3, where A' is a rotatory-reflexion about a single 4-al axis of the second sort. (Fig. 37).

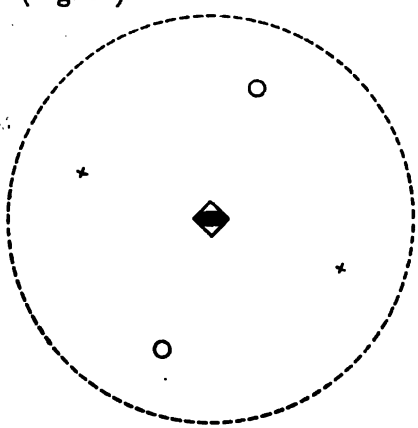

Fig. 37. C'_4.

§ 3. The rotation-axis of the groups C_2, C_3, C_4, C_6 is brought into self-coincidence by a reflexion S in a plane perpendicular

to the axis. We thus have four groups $C_{2h}, C_{3h}, C_{4h}, C_{6h}$ (Figs 38–41) whose operations are

$$\left.\begin{matrix} 1, & A, & A^2, & \ldots, & A^{n-1} \\ S, & A.S, & A^2.S, & \ldots, & A^{n-1}.S \end{matrix}\right\},$$

or, writing them in a different order,

$$\left.\begin{matrix} 1, & A, & A^2, & \ldots, & A^{n-1} \\ S, & S.A, & S.A^2, & \ldots, & S.A^{n-1} \end{matrix}\right\},$$

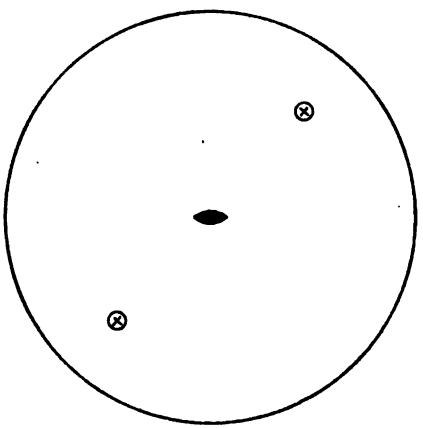

Fig. 38. C_{2h}.

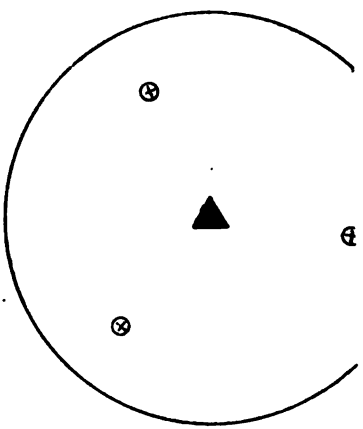

Fig. 39. C_{3h}.

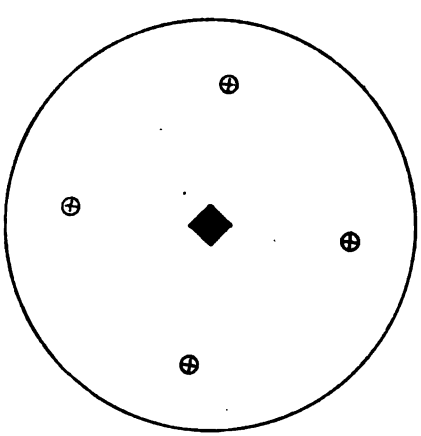

Fig. 40. C_{4h}.

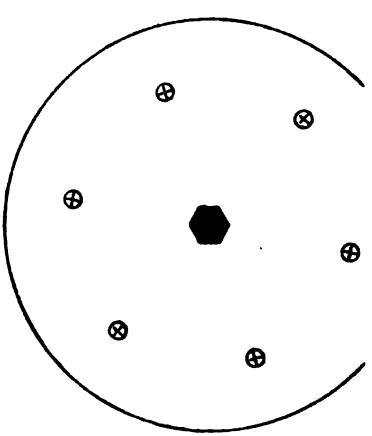

Fig. 41. C_{6h}.

where $n = 2$, 3, 4, or 6 (cf. chap. v, § 2, p. 52); and which contain one rotation-axis and a perpendicular plane of

symmetry. The groups C_{2h}, C_{4h}, C_{6h} contain a centre of symmetry.

§ 4. The rotation-axis of the groups C_2, C_3, C_4, C_6 is also brought into self-coincidence by a reflexion S_1 in a plane s_1 passing through the axis.

Let successive rotations through $\frac{2\pi}{n}$ about the n-al axis

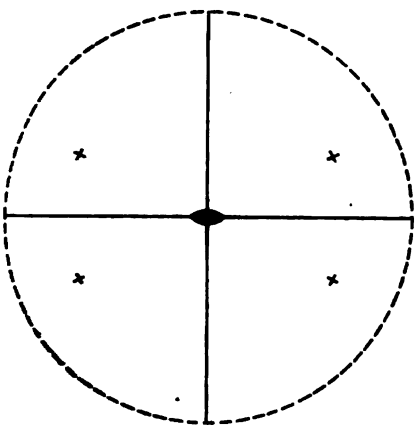

Fig. 42. C_{2v}.

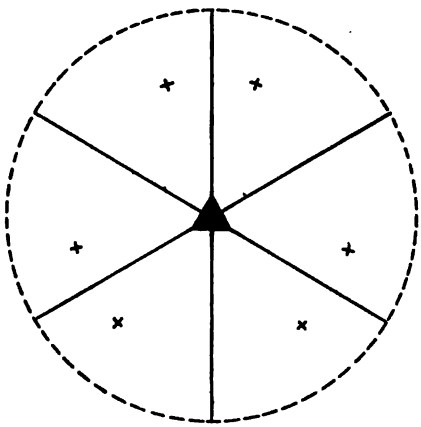

Fig. 43. C_{3v}.

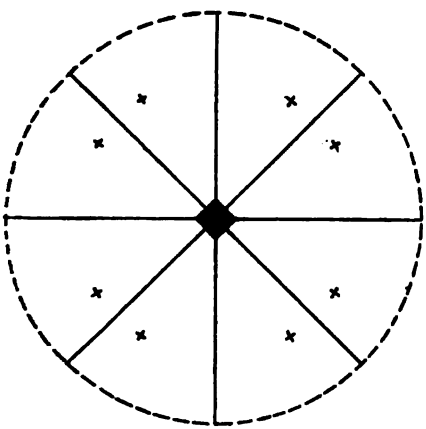

Fig. 44. C_{4v}.

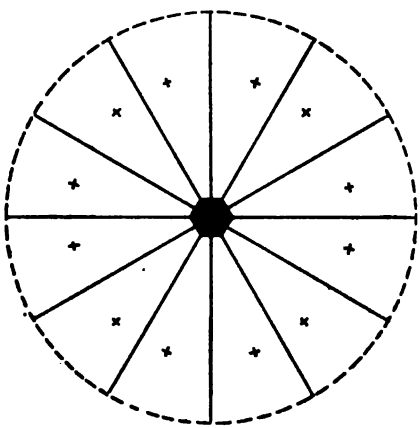

Fig. 45. C_{6v}.

of a group C_n bring s_1 into the positions s_1', s_1'' ..., $(n-1)$ in number if n is odd and $\left(\frac{n}{2}-1\right)$ in number if n is even; then these planes are also planes of symmetry. The rotation

through an angle $\frac{2\pi}{n}$ about the axis, followed by a reflexio in s_1, is equivalent to a reflexion in a plane σ_1 making a angle $\frac{\pi}{n}$ with s_1. (Cf. chap. iii. § 5, p. 26.)

If n is odd σ_1 coincides with one of the series $s_1, s_1', s_1'', \ldots$ but if n is even there is a series of $\frac{n}{2}$ symmetry-plan $\sigma_1, \sigma_1', \sigma_1'', \ldots$ belonging to the group and bisecting the angl between the planes $s_1, s_1', s_1'', \ldots$.

There can be no other plane of symmetry passing throug the axis. This may be proved by a method similar to th employed on p. 54.

It is readily seen that the planes $s_1, s_1', s_1'', \ldots$* are equiv lent, and so are the planes $\sigma_1, \sigma_1', \sigma_1'', \ldots$*, but that none the s_1^a are equivalent to any of the σ_1^a (if n is even).

The total number of symmetry-planes (whether n is od or even) is n.

The operations of the group are

$$\left.\begin{matrix} 1, & A, & A^2, & \ldots, & A^{n-1} \\ S_1, & A.S_1, & A^2.S_1, & \ldots, & A^{n-1}.S_1 \end{matrix}\right\}.$$

Taking $n = 2, 3, 4$, and 6 we have four groups C_{2v}, C_3 C_{4v}, C_{6v} (Figs. 42, 43, 44, 45.)

No reflexion can bring a single rotation-axis to self-coi cidence except those i planes passing throug or perpendicular to th axis.

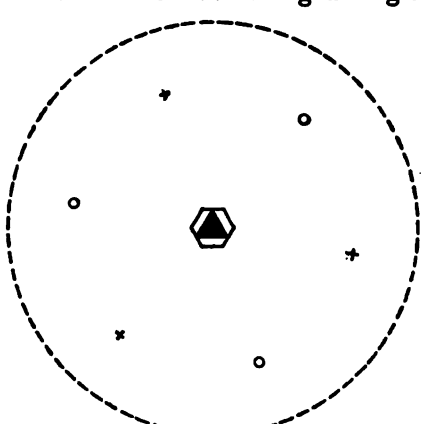

Fig. 46. C_{3i}.

§ 5. A centre of syn metry is an operatic bringing the axis of group C_n into self-coi cidence, but it creat only the group C_{nh} if n even.

If $n = 3$ we have a ne group C_{3i} (Fig. 46) whc operations are

$$\left.\begin{matrix} 1, & A, & A^2 \\ I, & A.I, & A^2.I \end{matrix}\right\}.$$

This group is that containing a 6-al axis of the second so (p. 36).

* These two series coincide if n is odd.

A rotatory-reflexion of angle $\frac{\pi}{2}$ only brings the axis of C_n into self-coincidence if its axis coincides with the axis of the rotatory-reflexion.

If $n = 2$ this leads us again to C_4'; we have already proved that if $n = 4$ or 6 we can obtain no new group applicable to crystallography by combining the operations of C_n with a rotatory-reflexion of angle $\frac{\pi}{2}$.

§ 6. A reflexion S in a plane perpendicular to the n-al axis of the group D_n brings the system of rotation-axes to self-

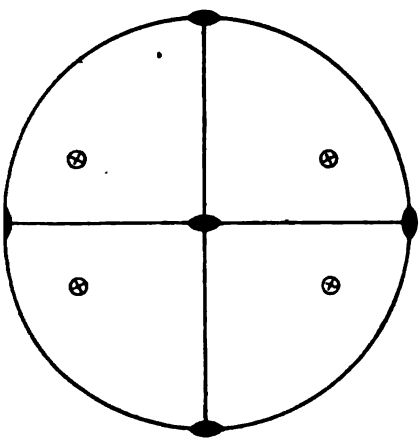

Fig. 47. D_{2h} or Q_h.

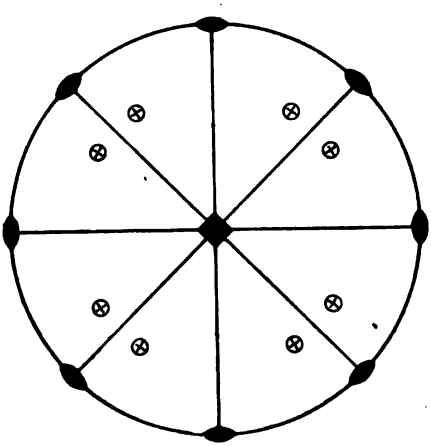

Fig. 48. D_{4h}.

coincidence; we get thus a group of the second sort D_{nh}; whose operations are

$1,$	$A,$	$A^2,$	...,	$A^{n-1};$
$S,$	$A.S,$	$A^2.S,$	...,	$A^{n-1}.S;$
$B,$	$A.B,$	$A^2.B,$	...,	$A^{n-1}.B.$
$B.S,$	$A.B.S,$	$A^2.B.S,$	...,	$A^{n-1}.B.S.$

If n is even the group D_{nh} must possess a centre of symmetry; in this case it must have also n planes of symmetry perpendicular to the n 2-al axes; each of these planes passes through the n-al axis and one of the 2-al axes.

Systems of equivalent points for the groups D_{2h}*, D_{4h}, D_{6h} are shown in Figs. 47, 48, and 49.

* Also called Q_h when we wish to call attention to the fact that all its axes have similar properties.

If n is odd we have no centre of symmetry, but since a 2-al axis and a symmetry-plane passing through it involve another symmetry-plane perpendicular to the first (chap. iii. § 5, p. 26), the group D_{3h} (Fig. 50) contains three symmetry-

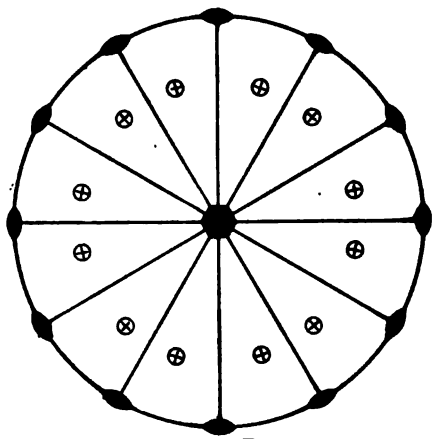

Fig. 49. D_{6h}.

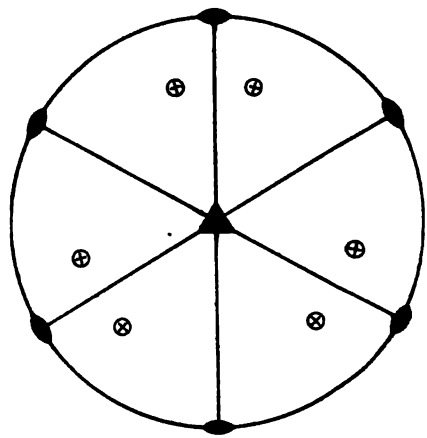

Fig. 50. D_{3h}.

planes each passing through the 3-al axis and one of the 2-al axes.

It may be seen that the n-al axis of the first sort of the group C_{nh} or D_{nh} is also an n-al axis of the second sort.

§ 7. The resultant of a reflexion in a plane s, followed by a rotation through π about an axis b making an angle a with s, is the resultant of a reflexion in s followed by a reflexion in the plane perpendicular to b (which makes an angle $\frac{\pi}{2} - a$ with s) and an inversion about the point O where b and s meet.

This is equivalent to a rotation through $(\pi - 2a)$ about an axis c lying in s and perpendicular to b followed by an inversion about O, i.e. to a rotation through $(\pi - 2a)$ about c followed by a rotation through π about c and a reflexion in the plane through O perpendicular to c, or to a rotatory-reflexion through an angle $-2a$ about c.

Hence if a group contains a plane of symmetry and a 2-al axis making an angle $\frac{\pi}{m}$ with it, it contains an m-al axis of the second sort lying in the plane of symmetry and perpendicular to the 2-al axis.

§ 8. A reflexion S_d in a symmetry-plane s_d bisecting the angles between the 2-al axes of the group D_n brings the system of axes of that group into self-coincidence; and hence creates a group of the second sort D_{nd} when combined with the rotations about these axes.

The operations of this group are

$1,$	$A,$	$A^2,$	...,	$A^{n-1};$
$S_d,$	$A.S_d,$	$A^2.S_d,$	...,	$A^{n-1}.S_d;$
$B,$	$A.B,$	$A^2.B,$	...,	$A^{n-1}.B.$
$B.S_d,$	$A.B.S_d,$	$A^2.B.S_d,$	...,	$A^{n-1}.B.S_d.$

By § 7 this group contains a $2n$-al axis of the second sort and therefore it is inconsistent with the law of rational indices unless $n = 2$ or 3.

Systems of equivalent points for these two cases are given in Figs. 51 and 52.

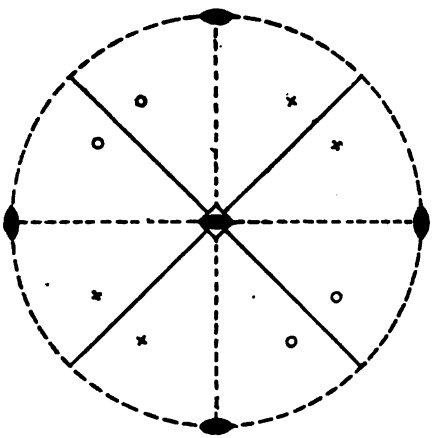

Fig. 51. D_{2d}.

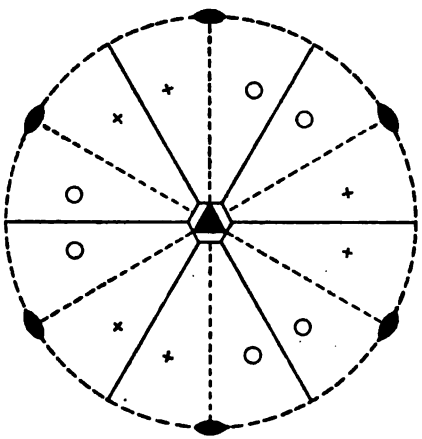

Fig. 52. D_{3d}.

In the case of D_{2d} the symmetry-planes are perpendicular to the 2-al axes, and there is therefore a centre of symmetry.

§ 9. A centre of inversion brings the axes of D_n into self-coincidence, but we get no *new* group by combining a centre of inversion with these axes.

A rotatory-reflexion of angle $\frac{\pi}{2}$ only brings the system of axes of D_n into self-coincidence, if its n-al axis coincides with the axis of rotatory-reflexion.

If $n = 2$ this leads us again to D_{2d}; we have already proved

that if $n = 4$ or 6 we can obtain no new group applicable to crystallography by combining the operations of D_n with a rotatory-reflexion of angle $\frac{\pi}{2}$.

§ 10. The system of axes of the group T is brought to self-coincidence by a reflexion S in a plane s passing through a pair of the 2-al axes.

The group formed by combining S with the operations of T must contain (p. 26) symmetry-planes through these two 2-al axes and perpendicular to the plane s; i.e. the group contains symmetry-planes perpendicular to each of the three 2-al axes, and has therefore a centre of symmetry.

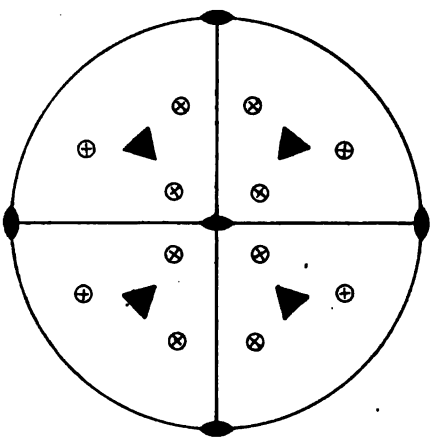

Fig. 53. T_h.

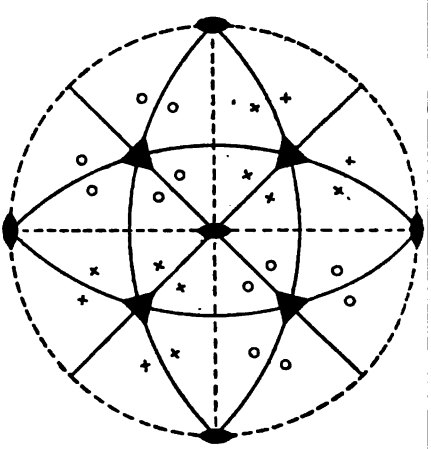

Fig. 54. T_d.

A system of equivalent points of this group T_h is shown in Fig. 53.

Exactly similarly we may derive a group of the second sort from the operations of T and a reflexion S_1 in a plane joining two adjacent 3-al axes.

Since a symmetry-plane passes through a 3-al rotation-axis the group must also possess two other symmetry-planes making angles of $\frac{\pi}{3}$ with this plane and passing through the same 3-al axis; hence the six planes which pass through any pair of the four 3-al axes must all be symmetry-planes.

This group is denoted by T_d (Fig. 54); it has no centre of symmetry.

Since there are only two groups of the second sort which can be derived from D_2, i.e. D_{2h} and D_{2d}, and since D_2 is a subgroup of T; only two groups of the second sort, i.e. T_h and T_d can be derived from T.

§ 11. For the same reason two groups of the second sort *at most* can be derived from O; and in fact the two groups obtained by combining the operations of O with S or S_1 are identical. For the group with the operation S has a centre of symmetry, and therefore this group has a reflexion in a plane perpendicular to any 2-al axis of O; i.e. it contains the operation S_1.

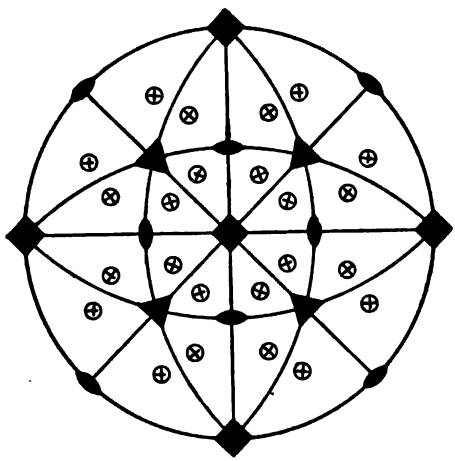

Fig. 55. O_h.

The operations of this group O_h (Fig. 55) may be written

$1,$	$U,$	$V,$	$W;$	
$R,$	$U.R,$	$V.R,$	$W.R;$	
$S,$	$U.S,$	$V.S,$	$W.S;$	
$R.S,$	$U.R.S,$	$V.R.S,$	$W.R.S;$	
$A,$	$U.A,$	$V.A,$	$W.A;$	
$A.R,$	$U.A.R,$	$V.A.R,$	$W.A.R;$	
$A.S,$	$U.A.S,$	$V.A.S,$	$W.A.S;$	
$A.R.S,$	$U.A.R.S,$	$V.A.R.S,$	$W.A.R.S;$	
$A^2,$	$U.A^2,$	$V.A^2,$	$W.A^2$ . . .	I
$A^2.R,$	$U.A^2.R,$	$V.A^2.R,$	$W.A^2.R$.	II
$A^2.S,$	$U.A^2.S,$	$V.A^2.S,$	$W.A^2.S$.	III
$A^2.R.S,$	$U.A^2.R.S,$	$V.A^2.R.S,$	$W.A^2.R.S$	IV

The operations of T_h are contained in lines I and III, those of T_d in lines I and IV.

It is important to notice that the operations of C_{2h}, D_{2h}, C_{4h}, D_{4h}, D_{3d}, C_{6h}, D_{6h}, T_h, O_h may be obtained by combining the operations of C_2, D_2, C_4, D_4, D_3, D_6, T, O respectively with an inversion. It would be better perhaps to denote D_{2h} (Q_h), T_h, O_h by the symbols Q_i, T_i, O_i respectively, but to avoid confusion Schoenflies' symbols are retained.

§ 12*. Only one group of the second sort can be derived from the group of § 6, chap. v; it is of course not applicable to crystallography. It has the fifteen planes joining opposite (and parallel) edges of the dodecahedron or icosahedron as symmetry-planes, and possesses a centre of symmetry.

* This section may be omitted without affecting the subsequent argument.

CHAPTER VII

THE COORDINATES OF EQUIVALENT POINTS.

§ 1. We have now proved that there are thirty-two and only thirty-two finite groups of movements which are consistent with the law of rational indices and are therefore applicable to crystallography.

We may arrange them as follows:—

I. 5 groups with more than one n-al axis ($n > 2$).
T, T_h, T_d, O, O_h ... Regular system.

II. 7 groups with a 6-al axis of the second or first sort.
C_{3i}, D_{3d}; C_6, C_{6h}, C_{6v}, D_6, D_{6h} ... Hexagonal system.

III. 7 groups with a 4-al axis of the second or first sort.
C'_4, D_{2d}; C_4, C_{4h}, C_{4v}, D_4, D_{4h} ... Tetragonal system.

IV. 5 groups with a 3-al axis of the first sort.
C_3, C_{3h}, C_{3v}, D_3, D_{3h} ... Trigonal system.

V. 6 groups with a 2-al axis of the second or first sort.
C_i; C_2, C_{2h}, C_{2v}, D_2, D_{2h} ... Digonal system.

VI. 2 groups with a 1-al axis of the second or first sort.
C_s; C_1 ... Monogonal system.

§ 2. For reasons which will appear later it is often more convenient to arrange these groups as follows*:—

I. Triclinic system† . . . . . C_1, C_i.

II. Monoclinic system‡ . . . . . C_s, C_2, C_{2h}.

III. Orthorhombic system . . . . . C_{2v}, D_2, D_{2h}.

IV. Tetragonal system
(*a*) First (sphenoidal) subdivision . . C'_4, D_{2d}.
(*b*) Second subdivision . C_4, C_{4h}, C_{4v}, D_4, D_{4h}.

* This is the more usual arrangement.
† Also called 'Anorthic system' or 'Asymmetric system.'
‡ Also called 'Monosymmetric system.'

V. Hexagonal system

(*a*) First (rhombohedral) subdivision $C_3, C_{3i}, C_{3v}, D_3, D_{3d}$.

(*b*) Second (sphenoidal) subdivision . . C_{3h}, D_{3h}.

(*c*) Third subdivision . . . $C_6, C_{6h}, C_{6v}, D_6, D_{6h}$.

VI. Regular system* . . . . . T, T_h, T_d, O, O_h.

The first subdivision of the hexagonal system is often treated as a separate system under the name 'Rhombohedral system'; and we shall so consider it in Part II.

We shall find in §§ 11 to 16 the coordinates of a system of equivalent points for each of these groups, choosing the axes of coordinates as conveniently as possible; the origin O in each case being at the centre of the crystal. We shall, however, first show how these coordinates may be obtained when the axes are chosen arbitrarily, and for that purpose shall first prove some general properties of coordinates and transformation of coordinates.

§ 3. Let OX, OY, OZ making angles α, β, γ with one another be taken as axes of coordinates; we denote by x, y, z the Cartesian coordinates of any point P.

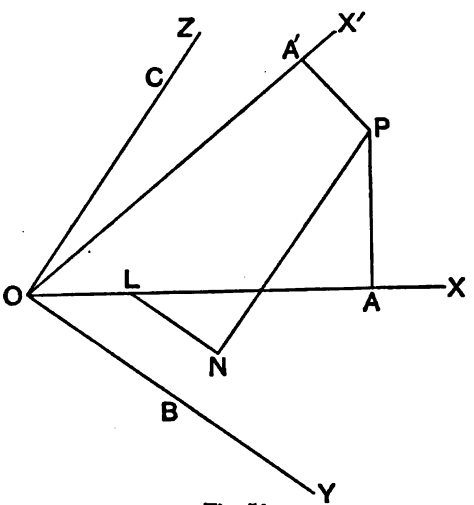

Fig. 56.

It is for some purposes convenient to use Fedorow coordinates.

Let planes through P perpendicular to the axes of x, y, and z meet these axes in A, B, and C respectively; then the lengths OA, OB, OC† are the *Fedorow coordinates* of the point P; we shall denote these coordinates by ξ, η, ζ.

They can be readily expressed in terms of x, y, z; for draw PN parallel to OZ to meet the plane AOB in N, and draw NL parallel to OY, meeting OX in L (Fig. 56).

* Also called 'Cubic system.'

† With the usual convention of sign.

Then (since the axes make angles α, β, γ with one another and PA is perpendicular to OX) we have, projecting the sides of the polygon $PNLA$ on OX, $LA = LN\cos\gamma + NP\cos\beta$; or since $LA = \xi - x$, $LN = y$, $NP = z$.

$$\text{Similarly} \quad \left.\begin{aligned} \xi &= x + y\cos\gamma + z\cos\beta \\ \eta &= x\cos\gamma + y + z\cos\alpha \\ \zeta &= x\cos\beta + y\cos\alpha + z \end{aligned}\right\} \quad \ldots\ldots \quad \text{(i).}$$

If the axes are mutually orthogonal $\xi = x$, $\eta = y$, $\zeta = z$; i. e. the Fedorow coordinates are identical with the Cartesian.

Solving for x, y, z we have

$$\left.\begin{aligned} x &= \frac{1}{\Delta}[\xi\sin^2\alpha + \eta(\cos\alpha\,.\cos\beta - \cos\gamma) \\ &\qquad + \zeta(\cos\gamma\,.\cos\alpha - \cos\beta)] \\ y &= \frac{1}{\Delta}[\xi(\cos\alpha\,.\cos\beta - \cos\gamma) + \eta\sin^2\beta \\ &\qquad + \zeta(\cos\beta\,.\cos\gamma - \cos\alpha)] \\ z &= \frac{1}{\Delta}[\xi(\cos\gamma\,.\cos\alpha - \cos\beta) \\ &\qquad + \eta(\cos\beta\,.\cos\gamma - \cos\alpha) + \zeta\sin^2\gamma] \end{aligned}\right\} \quad \text{(ii),}$$

where $\Delta \equiv (1 + 2\cos\alpha\,.\cos\beta\,.\cos\gamma - \cos^2\alpha - \cos^2\beta - \cos^2\gamma)$.

§ 4. If we wish to find the coordinates of the point P referred to any other axes OX', OY', OZ' making angles α', β', γ' with one another we proceed as follows.

Let x, y, z be the Cartesian coordinates of P referred to the old axes OX, OY, OZ; let x', y', z' be the Cartesian coordinates referred to the new axes OX', OY', OZ'; and let ξ, η, ζ; ξ', η', ζ' be the old and new Fedorow coordinates respectively.

Let the cosines of the angles which OX' makes with OX, OY, OZ be l_1, m_1, n_1 *; and let the corresponding quantities for OY' and OZ' be l_2, m_2, n_2 and l_3, m_3, n_3.

Draw PA' perpendicular to OX'; then $OA' = \xi'$.

Projecting the sides of the polygon $OLNPA'$ (Fig. 56) on OX' we get

$$\text{similarly} \quad \left.\begin{aligned} \xi' &= l_1x + m_1y + n_1z \\ \eta' &= l_2x + m_2y + n_2z \\ \zeta' &= l_3x + m_3y + n_3z \end{aligned}\right\} \quad \ldots\ldots \quad \text{(iii).}$$

* l_1, m_1, n_1 are connected by the relation

$$\Delta = l_1^2\sin^2\alpha + m_1^2\sin^2\beta + n_1^2\sin^2\gamma + 2m_1n_1(\cos\beta\,.\cos\gamma - \cos\alpha) + 2n_1l_1(\cos\gamma\,.\cos\alpha - \cos\beta) + 2l_1m_1(\cos\alpha\,.\cos\beta - \cos\gamma);$$

and so for l_2, m_2, n_2; l_3, m_3, n_3. This was proved in chap. ii, § 9 (p. 19).

In Fig. 56 interchange the letters A and A', B and B', C and C'; then projecting $OLNPA$ on OA we get

$$\left.\begin{aligned}\xi &= l_1x' + l_2y' + l_3z'\\ \text{similarly}\quad \eta &= m_1x' + m_2y' + m_3z'\\ \zeta &= n_1x' + n_2y' + n_3z'\end{aligned}\right\} \quad \ldots\ldots \text{(iv).}$$

§ 5. If we have equations of the form

$$\begin{aligned}x' &= p_1x + q_1y + r_1z\\ y' &= p_2x + q_2y + r_2z\\ z' &= p_3x + q_3y + r_3z,\end{aligned}$$

we call

$$\begin{vmatrix} p_1 & q_1 & r_1\\ p_2 & q_2 & r_2\\ p_3 & q_3 & r_3\end{vmatrix}$$

'the determinant of the transformation $(x'y'z';\ xyz)$.'

If we have also

$$\begin{aligned}x'' &= P_1x' + Q_1y' + R_1z'\\ y'' &= P_2x' + Q_2y' + R_2z'\\ z'' &= P_3x' + Q_3y' + R_3z',\end{aligned}$$

then

$$\begin{aligned}x'' &= (p_1P_1 + p_2Q_1 + p_3R_1)x + (q_1P_1 + q_2Q_1 + q_3R_1)y\\ &\qquad + (r_1P_1 + r_2Q_1 + r_3R_1)z\\ y'' &= (p_1P_2 + p_2Q_2 + p_3R_2)x + (q_1P_2 + q_2Q_2 + q_3R_2)y\\ &\qquad + (r_1P_2 + r_2Q_2 + r_3R_2)z\\ z'' &= (p_1P_3 + p_2Q_3 + p_3R_3)x + (q_1P_3 + q_2Q_3 + q_3R_3)y\\ &\qquad + (r_1P_3 + r_2Q_3 + r_3R_3)z,\end{aligned}$$

from which we readily verify that the determinant of the transformation $(x''y''z'';\ xyz)$ is the product of the determinants of the transformations

$$(x''y''z'';\ x'y'z') \quad \text{and} \quad (x'y'z';\ xyz).$$

The transformations $(xyz;\ x'y'z')$, $(x'y'z';\ xyz)$ are said to be reciprocal to one another; for instance the two transformations given by equations (i) and (ii) * are reciprocal.

Since the determinant of the transformation $(xyz;\ xyz)$,

$$\text{i. e.} \quad \begin{vmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix} = 1,$$

therefore the product of the determinants of two reciprocal transformations = 1.

* P. 75.

Thus for example since the determinant of $(\xi\eta\zeta;\ xyz)$ —equations (i)—is Δ, therefore the determinant of $(xyz;\ \xi\eta\zeta)$ —equations (ii)—is $\frac{1}{\Delta}$.

§ 6. Substituting in equations (iii) the values of x, y, z given by equations (ii), we have

$$\xi' = \frac{1}{\Delta}[\xi\{l_1\sin^2\alpha + m_1(\cos\alpha.\cos\beta - \cos\gamma) + n_1(\cos\gamma.\cos\alpha - \cos\beta)\}$$
$$+\eta\{l_1(\cos\alpha.\cos\beta - \cos\gamma) + m_1\sin^2\beta + n_1(\cos\beta.\cos\gamma - \cos\alpha)\}$$
$$+\zeta\{l_1(\cos\gamma.\cos\alpha - \cos\beta) + m_1(\cos\beta.\cos\gamma - \cos\alpha) + n_1\sin^2\gamma\}]$$

and two similar equations . . . . . (v).

The determinant of this transformation $(\xi'\eta'\zeta';\ \xi\eta\zeta)$ is the product of the determinants of $(\xi'\eta'\zeta';\ xyz)$ and $(xyz;\ \xi\eta\zeta)$, and is therefore equal to $\frac{D}{\Delta}$, where D denotes the determinant

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}.$$

Again from (ii)

$$x' = \frac{1}{\Delta'}[\xi'\sin^2\alpha' + \eta'(\cos\alpha'.\cos\beta' - \cos\gamma') + \zeta'(\cos\gamma'.\cos\alpha' - \cos\beta')]$$

and two similar equations; where

$$\Delta' \equiv (1 + 2\cos\alpha'.\cos\beta'.\cos\gamma' - \cos^2\alpha' - \cos^2\beta' - \cos^2\gamma').$$

Substituting in (iv) we find

$$\xi = \frac{1}{\Delta'}[\xi'\{l_1\sin^2\alpha' + l_2(\cos\alpha'.\cos\beta' - \cos\gamma') + l_3(\cos\gamma'.\cos\alpha' - \cos\beta')\}$$
$$+\eta'\{l_1(\cos\alpha'.\cos\beta' - \cos\gamma') + l_2\sin^2\beta' + l_3(\cos\beta'.\cos\gamma' - \cos\alpha')\}$$
$$+\zeta'\{l_1(\cos\gamma'.\cos\alpha' - \cos\beta') + l_2(\cos\beta'.\cos\gamma' - \cos\alpha') + l_3\sin^2\gamma'\}]$$

and two similar equations . . . . . (vi).

Hence the determinant of the transformation $(\xi\ \eta\ \zeta;\ \xi'\ \eta'\ \zeta')$

$$= \frac{\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}}{\Delta'} = \frac{D}{\Delta'}.$$

Another expression for ξ, η, ζ in terms of ξ', η', ζ' is found by solving for x, y, z from equations (iii) and substituting in (i).

We have then

$$\xi = \frac{1}{D}\left[\xi' \begin{vmatrix} 1 & \cos\gamma & \cos\beta \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} + \eta' \begin{vmatrix} l_1 & m_1 & n_1 \\ 1 & \cos\gamma & \cos\beta \\ l_3 & m_3 & n_3 \end{vmatrix} + \zeta' \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ 1 & \cos\gamma & \cos\beta \end{vmatrix}\right]$$

and two similar equations . . . . . (vii).

Again substituting from equations (iv) in (ii) we have

$$\begin{aligned} x = \frac{1}{\Delta}\big[&x'\{l_1\sin^2\alpha + m_1(\cos\alpha\,.\cos\beta - \cos\gamma) + n_1(\cos\gamma\,.\cos\alpha - \cos\beta)\} \\ &+ y'\{l_2\sin^2\alpha + m_2(\cos\alpha\,.\cos\beta - \cos\gamma) + n_2(\cos\gamma\,.\cos\alpha - \cos\beta)\} \\ &+ z'\{l_3\sin^2\alpha + m_3(\cos\alpha\,.\cos\beta - \cos\gamma) + n_3(\cos\gamma\,.\cos\alpha - \cos\beta)\}\big] \end{aligned}$$

and two similar equations . . . . . (viii).

The determinant of this transformation $(x\ y\ z;\ x'\ y'\ z')$ = the product of the determinants of $(x\,y\,z;\ \xi\ \eta\ \zeta)$ and $(\xi\ \eta\ \zeta;\ x'\ y'\ z') = \frac{D}{\Delta}$.

Similarly from the equations

$$x' = \frac{1}{\Delta}\left[\xi'\sin^2\alpha' + \eta'(\cos\alpha'.\cos\beta' - \cos\gamma') + \zeta'(\cos\gamma'.\cos\alpha' - \cos\beta')\right], \text{ &c.}$$

and (iii) we have

$$\begin{aligned} x' = \frac{1}{\Delta'}\big[&x\{l_1\sin^2\alpha' + l_2(\cos\alpha'.\cos\beta' - \cos\gamma') + l_3(\cos\gamma'.\cos\alpha' - \cos\beta')\} \\ &+ y\{m_1\sin^2\alpha' + m_2(\cos\alpha'.\cos\beta' - \cos\gamma') + m_3(\cos\gamma'.\cos\alpha' - \cos\beta')\} \\ &+ z\{n_1\sin^2\alpha' + n_2(\cos\alpha'.\cos\beta' - \cos\gamma') + n_3(\cos\gamma'.\cos\alpha' - \cos\beta')\}\big] \end{aligned}$$

and two similar equations . . . . . (ix).

The determinant of this transformation $(x'\ y'\ z';\ x\ y\ z)$ is evidently $\frac{D}{\Delta'}$.

Another expression for x', y', z' in terms of x, y, z is found by solving for x', y', z' from (iv), and substituting for ξ, η, ζ from (i).

We have then

$$x' = \frac{1}{D}\left[x\begin{vmatrix} 1 & l_2 & l_3 \\ \cos\gamma & m_2 & m_3 \\ \cos\beta & n_2 & n_3 \end{vmatrix} + y\begin{vmatrix} \cos\gamma & l_2 & l_3 \\ 1 & m_2 & m_3 \\ \cos\alpha & n_2 & n_3 \end{vmatrix} + z\begin{vmatrix} \cos\beta & l_2 & l_3 \\ \cos\alpha & m_2 & m_3 \\ 1 & n_2 & n_3 \end{vmatrix}\right]$$

and two similar equations . . . . . (x).

It is interesting to compare equations (v) and (vi) with equations (viii) and (ix); for other forms of the equations (v), (vii), (viii), and (x) see note I at the end of this chapter.

Since the product of the determinants of the two transformations $(x'y'z' ; xyz)$ and $(xyz ; x'y'z') = 1$,

$$\therefore \quad \frac{D}{\Delta'} \cdot \frac{D}{\Delta} = 1 \quad \text{or} \quad D^2 = \Delta\Delta' *\dagger.$$

An interesting case is that for which the determinant of $(x'y'z' ; xyz)$, and therefore the determinants of $(xyz ; x'y'z')$, $(\xi'\eta'\zeta' ; \xi\eta\zeta)$, $(\xi\eta\zeta ; \xi'\eta'\zeta') = \pm 1$.

Then $D^2 = \Delta'^2$, and $\therefore \Delta = \Delta'$.

Since $\Delta \equiv 1 - (\cos\gamma - \cos\alpha \,.\, \cos\beta)^2 - \cos^2\alpha \,.\, \sin^2\beta - \cos^2\beta$, we cannot have $\Delta = \Delta' = 1$ unless

$$\cos\alpha = \cos\beta = \cos\gamma = \cos\alpha' = \cos\beta' = \cos\gamma' = 0 \ddagger,$$

i.e. both sets of axes are mutually orthogonal; if however $\Delta = \Delta' \neq 1$ it does not necessarily *follow* that

$$\alpha' = \alpha, \quad \beta' = \beta, \quad \gamma' = \gamma,$$

though this is quite *consistent* with $\Delta = \Delta'$.

If $\alpha' = \alpha$, $\beta' = \beta$, $\gamma' = \gamma$ the set of axes OX', OY', OZ' is either congruent or enantiomorphous to the set OX, OY, OZ.

§ 7. We can obtain a criterion as to the sign of the determinant of transformation in the general case by the following reasoning.

* For neither Δ nor Δ' can = 0. For if e.g.

$$1 + 2\cos\alpha \,.\, \cos\beta \,.\, \cos\gamma - \cos^2\alpha - \cos^2\beta - \cos^2\gamma = 0,$$

solving for $\cos\gamma$ we see that $\pm\gamma = \alpha \pm \beta \pm$ a multiple of 2π; and therefore the lines OX, OY, OZ are coplanar contrary to hypothesis. The relation $D^2 = \Delta\Delta'$ can also be proved by considering the product of the determinants of $(\xi'\eta'\zeta' ; \xi\eta\zeta)$ and $(\xi\eta\zeta ; \xi'\eta'\zeta')$.

† This is the relation between the cosines of the angles which any six straight lines make with one another. See note II at the end of this chapter.

‡ The angles $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ are of course considered all *real*.

Suppose the original axes OX, OY, OZ fixed, while the positions of the new axes OX', OY', OZ' are varied *continuously*; then the determinant of any one of the transformations of § 6 will vary also *continuously*, and therefore cannot change its sign without passing through the value 0 or ∞. This can only happen when $\Delta' = 0$*, i.e. if OX', OY', OZ' become coplanar; for the determinant in question has the value $\left(\frac{\Delta'}{\Delta}\right)^{\pm\frac{1}{2}}$. Hence if the positions of OX', OY', OZ' are varied continuously in such a way that they never become coplanar, the *sign* of $\sqrt{\frac{\Delta'}{\Delta}}$ (and of $\sqrt{\frac{\Delta}{\Delta'}}$) remains unaltered.

First suppose the set OX', OY', OZ' congruent to the set OX, OY, OZ; then OX' can be brought to coincide with OX, OY' with OY, OZ' with OZ by a continuous variation of position of the kind considered. Therefore the sign of the determinant of transformation from OX, OY, OZ to OX', OY', OZ' is the same as that of the transformation from OX, OY, OZ to OX, OY, OZ, i.e. the same as that of $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ and the determinant is positive. Now in the case considered $\Delta' = \Delta$, and therefore the determinant of transformation from a set OX, OY, OZ to a *congruent* set $OX', OY', OZ' = +1$.

Similarly if the set OX', OY', OZ' is enantiomorphous to the set OX, OY, OZ; OX' can be brought into coincidence with OX, OY' with OZ, OZ' with OY by a continuous change of position of the kind considered. Hence the sign of the determinant of transformation from OX, OY, OZ to OX', OY', OZ' is that of $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$, and the determinant is negative.

Also again $\Delta' = \Delta$, and therefore the determinant of transformation from a set OX, OY, OZ to an *enantiomorphous* set $OX', OY', OZ' = -1$.

Now let OX', OY', OZ' have any non-coplanar positions.

In the plane $X'OY'$ draw OE' perpendicular to OX', and on the same side of OX' (produced) as OY'; similarly draw OF' perpendicular to OX' in the plane $X'OZ'$ on the same side of OX' as OZ'. Let OE, OF have the same positions with regard to OX, OY, OZ as OE', OF' have with regard to OX', OY', OZ'.

Now let OY', OZ' be moved in the planes $X'OY'$, $X'OZ'$ respectively (so that they remain on the same side of OX'

* Assuming $\Delta \neq 0$; i. e. assuming OX, OY, OZ not coplanar.

as OE', OF') until the angle $X'OY' = XOY$ and the angle $X'OZ' = XOZ$.

Now let OE', OY' be rotated together about OX' so that OE', OF' never make an angle of 0 or π with one another until the angle $E'OF'$ measured in a positive direction equals EOF or (if that is impossible) $2\pi - EOF$ (measuring EOF in the same direction). Then in the former case the determinant of transformation from OX, OY, OZ to the altered positions of OX', OY', OZ' is $+1$, and therefore the determinant of transformation from OX, OY, OZ to the original positions of OX', OY', OZ' is positive; similarly in the latter case (when the altered value of $E'OF' = 2\pi - EOF$) the determinant of transformation from OX, OY, OZ to the altered positions of OX', OY', OZ' is -1, and therefore the determinant of transformation from OX, OY, OZ to the original positions of OX', OY', OZ' is negative.

§ 8. Now let the set OX', OY', OZ' be obtained from any congruent set OX, OY, OZ by rotating the latter through an angle θ about an axis ON.

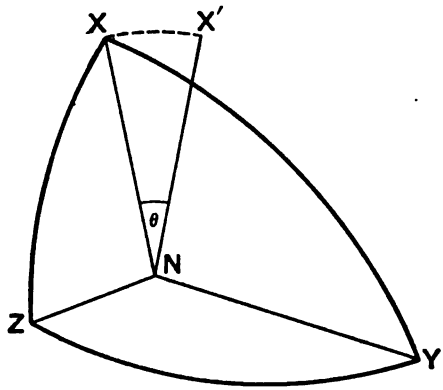

Fig. 57.

Let a sphere with centre O meet OX, OY, OZ, OX', OY', OZ', ON in X, Y, Z, X', Y', Z', N respectively; let the arcs $NX = NX' = \lambda$, $NY = NY' = \mu$, $NZ = NZ' = \nu$, and let

$$\cos\lambda = l,\ \cos\mu = m,\ \cos\nu = n.$$

Then we have

$$YZ = \alpha,\ ZX = \beta,\ XY = \gamma,\ X'NX = \theta,$$
$$\cos XX' = l_1,\ \cos YX' = m_1,\ \cos ZX' = n_1$$

(Fig. 57; the notation of § 3 is used).

Then $\cos ZNY = \dfrac{\cos a - \cos\mu \,.\, \cos\nu}{\sin\mu \,.\, \sin\nu}$, &c.

and therefore

$$\sin ZNY = + \frac{\sqrt{\sin^2 a - m^2 - n^2 + 2\cos a \,.\, mn}}{\sin\mu \,.\, \sin\nu}, \text{ \&c.}$$

From the spherical triangle XNX',

$$l_1 = \cos^2\lambda + \sin^2\lambda \,.\, \cos\theta = l^2(1-\cos\theta) + \cos\theta.$$

From the triangle $X'NY$

$$m_1 = \cos\lambda \,.\, \cos\mu + \sin\lambda \,.\, \sin\mu \,.\, \cos(YNX - \theta) = lm(1-\cos\theta) + \cos\theta \,.\, \cos\gamma + \sin\theta\sqrt{\sin^2\gamma - l^2 - m^2 + 2\cos\gamma \,.\, lm}.$$

From the triangle $X'NZ$

$$n_1 = \cos\lambda \,.\, \cos\nu + \sin\lambda \,.\, \sin\nu \,.\, \cos(XNZ + \theta) = ln(1-\cos\theta) + \cos\theta \,.\, \cos\beta - \sin\theta\sqrt{\sin^2\beta - l^2 - n^2 + 2\cos\beta \,.\, ln}.$$

We have similar expressions for l_2, m_2, n_2; l_3, m_3, n_3.

Hence if we are given the coordinates of any point referred to any set of axes, we can obtain the coordinates of the point referred to the set obtained by rotating the original set through a known angle about a given axis. For all we require is the cosines of the angles which the new axes make with the old (§ 6), and these we have now shown how to obtain.

Now let OX, OY, OZ be brought to OX', OY', OZ' by some known operation of the second sort.

Produce XO, YO, ZO to X'', Y'', Z'' respectively; then OX, OY, OZ are brought into coincidence with OX'', OY'', OZ'' by an inversion about O, and OX'', OY'', OZ'' are brought to OX', OY', OZ' by some known operation of the first sort, i. e. by some known rotation. The above investigation gives us the cosines of the angles which OX', OY', OZ' make with OX'', OY'', OZ'', and by multiplying these by -1 we obtain the cosines of the angles which OX', OY', OZ' make with OX, OY, OZ. Hence we obtain the coordinates of any point referred to OX', OY', OZ' in terms of the coordinates referred to OX, OY, OZ.

§ 9. A useful case is that in which the set OX', OY', OZ' is the reflexion of the set OX, OY, OZ in a plane the normal to which makes angles with OX, OY, OZ whose cosines are l, m, n respectively.

In this case putting $\theta = \pi$ in the equations of § 8 and

changing the signs of the right-hand sides of those equations, we have

$$\left.\begin{array}{lll} l_1 = 1-2l^2, & m_1 = \cos\gamma - 2lm, & n_1 = \cos\beta - 2nl \\ l_2 = \cos\gamma - 2lm, & m_2 = 1-2m^2, & n_2 = \cos\alpha - 2mn \\ l_3 = \cos\beta - 2nl, & m_3 = \cos\alpha - 2mn, & n_3 = 1-2n^2. \end{array}\right\}.$$

§ 10. *To find the coordinates of all the points equivalent to any point P in any group of movements.*

Suppose we take any fixed reference-axes OX, OY, OZ, and suppose the coordinates of the point P with reference to these axes to be known.

Let the operation L of the group bring P to coincide with the equivalent point P'.

Then if L brings the reference-axes into the positions OX', OY', OZ', the coordinates of the point P' referred to OX', OY', OZ' are evidently the same as the known coordinates of P referred to OX, OY, OZ; and we have shown in the previous sections how to express the coordinates of P' referred to OX, OY, OZ in terms of its coordinates referred to OX', OY', OZ', when OX, OY, OZ are brought into coincidence with OX', OY', OZ' by any operation of the first or second sort.

If, however, we choose our reference-axes in each case as conveniently as possible, we can almost always write down the coordinates of a system of equivalent points by inspection.

Take, for instance, the case of D_{2d} whose operations are $1, A; B, A.B; S_d, A.S_d; B.S_d, A.B.S_d$ (p. 69). Taking the axes of z, y as the axes of A and B respectively, and the plane $x = y$ as the plane of S_d; we see that the point (xyz) is brought to coincide with $(\bar{x}\bar{y}z)$ by A and with $(\bar{x}y\bar{z})$ by B: similarly $(\bar{x}\bar{y}z)$ is brought to $(x\bar{y}\bar{z})$ by B, and therefore (xyz) is brought to $(x\bar{y}\bar{z})$ by $A.B$.

Again $(xyz), (\bar{x}\bar{y}z), (\bar{x}y\bar{z}), (x\bar{y}\bar{z})$ are brought respectively to $(yxz), (\bar{y}\bar{x}z), (y\bar{x}\bar{z}), (\bar{y}x\bar{z})$ by S_d, and therefore (xyz) is brought to $(yxz), (\bar{y}\bar{x}z), (y\bar{x}\bar{z}), (\bar{y}x\bar{z})$ by $S_d, A.S_d, B.S_d, A.B.S_d$. The system of equivalent points of D_{2d} is therefore that given in § 14.

In the following sections we shall point out which reference-axes are the most suitable, and leave the reader to verify the conclusions stated.

§ 11. In the case of the groups C_1, C_2 belonging to the triclinic system (§ 2), there is no line specially pointed out by the symmetry of the groups as a proper line to be taken as an axis of coordinates. We take therefore *any* three lines as

axes (in general no two of these lines are at right angles to one another—hence the name 'triclinic'); let xyz be the Cartesian coordinates of any point P and let $\xi\eta\zeta$ be its Fedorow coordinates, then the system of points equivalent to (and including) P is

for	C_1 ..... xyz.	$\xi\eta\zeta$
and for	C_2 ... $xyz, \bar{x}\bar{y}\bar{z}$.	$\xi\eta\zeta, {}^{-}\xi{}^{-}\eta{}^{-}\zeta$*.

§ 12. In the case of C_s, C_2, C_{2h} belonging to the monoclinic system, the line perpendicular to the symmetry-plane, or parallel to the 2-al rotation-axis is pointed out by the symmetry of the groups as a suitable line for one of the coordinate axes. Take then this line as the axis of z, and *any* two lines perpendicular to it (not in general at right angles to each other—hence the name 'monoclinic') as axes of x and y.

Then a system of equivalent points is

for	C_s ... $xyz, xy\bar{z}$.	$\xi\eta\zeta, \xi\eta{}^{-}\zeta$.
for	C_2 ... $xyz, \bar{x}\bar{y}z$.	$\xi\eta\zeta, {}^{-}\xi{}^{-}\eta\zeta$.
and for	C_{2h} ... $xyz, \bar{x}\bar{y}z, xy\bar{z}, \bar{x}\bar{y}\bar{z}$.	$\xi\eta\zeta, {}^{-}\xi{}^{-}\eta\xi, \xi\eta{}^{-}\zeta, {}^{-}\xi{}^{-}\eta{}^{-}\xi$.

§ 13. For the group C_{2v} we take the 2-al rotation-axis as axis of z, and the two lines perpendicular to it and lying in the symmetry-planes as axes of x and y; for the other two groups of the orthorhombic system, D_2 and D_{2h}, we take the three 2-al symmetry-axes as coordinate axes.

In each case the three coordinate axes are mutually orthogonal; the Cartesian and Fedorow coordinates are therefore identical. We have as the systems of Cartesian coordinates

for	C_{2v} ... $xyz, \bar{x}\bar{y}z, x\bar{y}z, \bar{x}yz$.
for	D_2 ... $xyz, \bar{x}\bar{y}z, \bar{x}y\bar{z}, x\bar{y}\bar{z}$.
for	D_{2h} ... $xyz, \bar{x}\bar{y}z, \bar{x}y\bar{z}, x\bar{y}\bar{z}$; $xy\bar{z}, \bar{x}\bar{y}\bar{z}, \bar{x}yz, x\bar{y}z$.

and the Fedorow coordinates are identical with these. We notice that every symmetry-operation of any group of the triclinic, monoclinic, or orthorhombic systems brings each reference-axis into coincidence with itself.

§ 14. In the tetragonal system we again take the three coordinate axes as mutually orthogonal.

The z axis is the 4-al symmetry-axis of the second sort for C_4' and D_{2d}, and the 4-al rotation-axis for C_4, C_{4h}, C_{4v}, D_4, D_{4h}.

The axes of x and y are 2-al rotation-axes for D_{2d}, D_4, and D_{4h}, lie in symmetry-planes for C_{4v}, and are any two lines

* $\bar{x}$ is printed for $-x$, ${}^{-}\xi$ for $-\xi$, &c. $-\xi$ would be *written* $\bar{\xi}$, &c.

perpendicular to each other and to the z axis in the case of C_4, C_4', and C_{4h}.

The systems of Cartesian coordinates are

for C_4' ... xyz, $\bar{y}x\bar{z}$, $\bar{x}\bar{y}z$, $y\bar{x}\bar{z}$.

for D_{2d} ... xyz, $\bar{x}\bar{y}z$, $\bar{x}y\bar{z}$, $x\bar{y}\bar{z}$; yxz, $\bar{y}\bar{x}z$, $y\bar{x}\bar{z}$, $\bar{y}x\bar{z}$.

for C_4 ... xyz, $\bar{y}xz$, $\bar{x}\bar{y}z$, $y\bar{x}z$.

for C_{4h} ... xyz, $\bar{y}xz$, $\bar{x}\bar{y}z$, $y\bar{x}z$; $xy\bar{z}$, $\bar{y}x\bar{z}$, $\bar{x}\bar{y}\bar{z}$, $y\bar{x}\bar{z}$.

for C_{4v} ... xyz, $\bar{y}xz$, $\bar{x}\bar{y}z$, $y\bar{x}z$; yxz, $x\bar{y}z$, $\bar{y}\bar{x}z$, $\bar{x}yz$.

for D_4 ... xyz, $\bar{y}xz$, $\bar{x}\bar{y}z$, $y\bar{x}z$; $\bar{x}y\bar{z}$, $yx\bar{z}$, $x\bar{y}\bar{z}$, $\bar{y}\bar{x}\bar{z}$.

for D_{4h} ... xyz, $\bar{y}xz$, $\bar{x}\bar{y}z$, $y\bar{x}z$; $\bar{x}y\bar{z}$, $yx\bar{z}$, $x\bar{y}\bar{z}$, $\bar{y}\bar{x}\bar{z}$;
$xy\bar{z}$, $\bar{y}x\bar{z}$, $\bar{x}\bar{y}\bar{z}$, $y\bar{x}\bar{z}$; $\bar{x}yz$, yxz, $x\bar{y}z$, $\bar{y}\bar{x}z$.

The Fedorow coordinates are identical with these.

§ 15. In the hexagonal system there are various methods of choosing the coordinate axes.

(1) Sometimes they are taken mutually orthogonal as in the orthorhombic and tetragonal systems, we shall not however discuss this method.

(2) A method often employed in the case of the first subdivision of the hexagonal system is the following:—As the axis of z is chosen, for the groups C_3 and C_{3i} any line, for the groups C_{3v} and D_{3d} any line lying in a symmetry-plane, and for the group D_3 (and D_{3d}) any line lying in a plane perpendicular to a 2-al rotation-axis. For the axes of x and y are taken the positions into which the axis of z is brought by rotations through $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ respectively about the 3-al rotation-axis, so that the three coordinate axes are situated symmetrically with respect to the 3-al axis.

Systems of equivalent points are then

for C_3 ... xyz, zxy, yzx.

for C_{3i} ... xyz, zxy, yzx; $\bar{x}\bar{y}\bar{z}$, $\bar{z}\bar{x}\bar{y}$, $\bar{y}\bar{z}\bar{x}$.

for C_{3v} ... xyz, zxy, yzx; yxz, xzy, zyx.

for D_3 ... xyz, zxy, yzx; $\bar{y}\bar{x}\bar{z}$, $\bar{x}\bar{z}\bar{y}$, $\bar{z}\bar{y}\bar{x}$.

for D_{3d} ... xyz, zxy, yzx; $\bar{y}\bar{x}\bar{z}$, $\bar{x}\bar{z}\bar{y}$, $\bar{z}\bar{y}\bar{x}$;
yxz, xzy, zyx; $\bar{x}\bar{y}\bar{z}$, $\bar{z}\bar{x}\bar{y}$, $\bar{y}\bar{z}\bar{x}$.

We can obtain the systems of Fedorow coordinates by putting ξ, η, ζ for x, y, z in the above.

This method is, however, not so suitable for the second and third subdivisions of the system, and in what follows we shall use method

(3) In this case we refer any point to four coordinate axes, OX_1, OX_2, OX_3 and OZ. The axis of z is in each case the 3-al

or 6-al rotation-axis, and the three remaining axes make angles of $\frac{2\pi}{3}$ with one another and are all perpendicular to the axis of z.

We denote the Fedorow coordinates of any point P by $\xi_1, \xi_2, \xi_3, \zeta$; the ordinary definition of Cartesian coordinates is not applicable to the case of more than three reference-axes.

The quantities $\xi_1, \xi_2, \xi_3, \zeta$ must be connected by one relation, we may find it as follows:—

Draw perpendiculars PE_1, PE_2, PE_3 to OX_1, OX_2, OX_3, and PP' perpendicular to the plane of these three axes (Fig. 58); then $P'E_1$ is perpendicular to OE_1, $P'E_2$ to OE_2, and $P'E_3$ to OE_3.

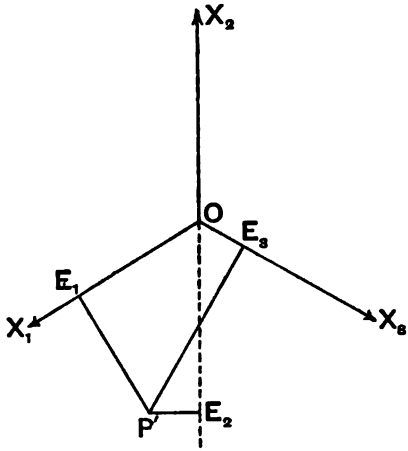

Fig. 58.

Now the angle

$$P'OX_2 - P'OX_1 = \frac{2\pi}{3},$$

and

$$P'OX_2 + P'OX_3 = \frac{4\pi}{3};$$

adding these two equations we have $P'OX_2$

$$= \pi - \left(\frac{P'OX_1 - P'OX_3}{2}\right),$$

subtracting we have

$$\left(\frac{P'OX_1 + P'OX_3}{2}\right) = \frac{\pi}{3}.$$

$$\therefore \quad OP'(\cos P'OX_1 + \cos P'OX_2 + \cos P'OX_3)$$

$$= OP'\left[2\cos\left(\frac{P'OX_1 + P'OX_3}{2}\right)\cos\left(\frac{P'OX_1 - P'OX_3}{2}\right)\right.$$

$$\left. + \cos P'OX_2\right] = OP'\left(-2\cos\frac{\pi}{3}\cdot\cos P'OX_2 + \cos P'OX_2\right) = 0,$$

and therefore $OE_1 + OE_2 + OE_3 = 0$.

Hence

$$\xi_1 + \xi_2 + \xi_3 = 0, \text{ for } OE_1 = \xi_1,\ OE_2 = \xi_2,\ OE_3 = \xi_3.$$

As the position of OX_1 we take, for C_3, C_{3i}, C_{3h}, C_6, C_{6h} any line perpendicular to the symmetry-axis, for C_{3v}, C_{6v} a line in a symmetry-plane perpendicular to the rotation-axis, and for D_3, D_{3d}, D_{3h}, D_6, D_{6h} a 2-al symmetry-axis.

The Fedorow coordinates of an equivalent system are then

for D_{6h}

$\xi_1\xi_2\xi_3\zeta,$	$\xi_3\xi_1\xi_2\zeta,$	$\xi_2\xi_3\xi_1\zeta$ . . .	I.
$-\xi_1-\xi_2-\xi_3\zeta,$	$-\xi_3-\xi_1-\xi_2-\zeta,$	$-\xi_2-\xi_3-\xi_1\zeta$ . .	II.
$\xi_3\xi_2\xi_1-\zeta,$	$\xi_2\xi_1\xi_3-\zeta,$	$\xi_1\xi_3\xi_2-\zeta$ . . .	III.
$-\xi_3-\xi_2-\xi_1-\zeta,$	$-\xi_2-\xi_1-\xi_3-\zeta,$	$-\xi_1-\xi_3-\xi_2-\zeta$.	IV.
$\xi_1\xi_2\xi_3-\zeta,$	$\xi_3\xi_1\xi_2-\zeta,$	$\xi_2\xi_3\xi_1-\zeta$ . . .	V.
$-\xi_1-\xi_2-\xi_3-\zeta,$	$-\xi_3-\xi_1-\xi_2-\zeta,$	$-\xi_2-\xi_3-\xi_1-\zeta$.	VI.
$\xi_3\xi_2\xi_1\zeta,$	$\xi_2\xi_1\xi_3\zeta,$	$\xi_1\xi_3\xi_2\zeta$ . . .	VII.
$-\xi_3-\xi_2-\xi_1\zeta,$	$-\xi_2-\xi_1-\xi_3\zeta,$	$-\xi_1-\xi_3-\xi_2\zeta$ . .	VIII.

for C_3 the 1st of the above lines,
for C_{3i} the 1st and 6th of the above lines,
for C_{3v} the 1st and 7th of the above lines,
for D_3 the 1st and 3rd of the above lines,
for D_{3d} the 1st, 3rd, 6th and 8th of the above lines,
for C_{3h} the 1st and 5th of the above lines,
for D_{3h} the 1st, 3rd, 5th and 7th of the above lines,
for C_6 the 1st and 2nd of the above lines,
for C_{6h} the 1st, 2nd, 5th and 6th of the above lines,
for C_{6v} the 1st, 2nd, 7th and 8th of the above lines,
for D_6 the 1st, 2nd, 3rd and 4th of the above lines.

In each case the relation $\xi_1+\xi_2+\xi_3=0$ holds.

We notice that every symmetry-operation of any group of the tetragonal or hexagonal systems brings the axis of z into self-coincidence; and brings any one of the other reference-axes into coincidence with itself or some other reference-axis perpendicular to the axis of z.

§ 16. For the regular system we take the three 2-al symmetry-axes of T, T_h, and T_d; or the three 4-al symmetry-axes of O and O_h as axes of reference. They are mutually perpendicular so that the Fedorow coordinates coincide with the Cartesian.

We have for O_h the system

$xyz, x\bar{y}\bar{z}, \bar{x}y\bar{z}, \bar{x}\bar{y}z$; $zxy, \bar{z}x\bar{y}, \bar{z}\bar{x}y, z\bar{x}\bar{y}$; $yzx, \bar{y}\bar{z}x, y\bar{z}\bar{x}, \bar{y}z\bar{x}$	I.
$yxz, \bar{y}xz, y\bar{x}z, \bar{y}\bar{x}z$; $xzy, x\bar{z}y, \bar{x}\bar{z}y, \bar{x}zy$; $zyx, \bar{z}\bar{y}x, \bar{z}yx, z\bar{y}x$	II.
$xy\bar{z}, x\bar{y}z, \bar{x}yz, \bar{x}\bar{y}\bar{z}$; $zx\bar{y}, \bar{z}xy, \bar{z}\bar{x}\bar{y}, z\bar{x}y$; $yz\bar{x}, \bar{y}\bar{z}\bar{x}, y\bar{z}x, \bar{y}zx$	III.
$yxz, \bar{y}x\bar{z}, y\bar{x}\bar{z}, \bar{y}\bar{x}z$; $xzy, x\bar{z}\bar{y}, \bar{x}\bar{z}y, \bar{x}z\bar{y}$; $zyx, \bar{z}\bar{y}x, \bar{z}y\bar{x}, z\bar{y}\bar{x}$	IV.

for T the 1st of the above lines,
for T_h the 1st and 3rd of the above lines,
for T_d the 1st and 4th of the above lines,
for O the 1st and 2nd of the above lines.

We notice that every symmetry-operation of any group of the regular system brings any reference-axis into coincidence with itself or one of the other reference-axes.

NOTE I.

Some of the equations of § 6 may be put into a simpler form by using the *direction-ratios* L_1, M_1, N_1; L_2, M_2, N_2; L_3, M_3, N_3, of OX', OY', OZ' referred to the axes OX, OY, OZ, instead of the cosines of the angles which OX', OY', OZ' make with these axes. We have

$$\begin{aligned} l_1 &= L_1 + M_1 \cos\gamma + N_1 \cos\beta \\ m_1 &= L_1 \cos\gamma + M_1 + N_1 \cos\alpha \\ n_1 &= L_1 \cos\beta + M_1 \cos\alpha + N_1 \end{aligned}$$

and two similar sets of equations, by a method similar to that used in obtaining equations (i) of § 3.

Substituting for l_1, m_1, n_1, &c., in equations (v) and (viii), we have

$\xi' = \xi L_1 + \eta M_1 + \zeta N_1$, and two similar equations . (v)′,

and

$x = x' L_1 + y' L_2 + z' L_3$ and two similar equations . (viii)′.

(v)′ can also be proved by direct application of (iii).

Solving, we have

$$\xi = \frac{1}{\begin{vmatrix} L_1 & M_1 & N_1 \\ L_2 & M_2 & N_2 \\ L_3 & M_3 & N_3 \end{vmatrix}} \left[\xi' \begin{vmatrix} M_2 & N_2 \\ M_3 & N_3 \end{vmatrix} + \eta' \begin{vmatrix} M_3 & N_3 \\ M_1 & N_1 \end{vmatrix} + \zeta' \begin{vmatrix} M_1 & N_1 \\ M_2 & N_2 \end{vmatrix} \right]$$

and two similar equations . . . . . (vii)′;

$$x' = \frac{1}{\begin{vmatrix} L_1 & M_1 & N_1 \\ L_2 & M_2 & N_2 \\ L_3 & M_3 & N_3 \end{vmatrix}} \left[x \begin{vmatrix} M_2 & M_3 \\ N_2 & N_3 \end{vmatrix} + y \begin{vmatrix} N_2 & N_3 \\ L_2 & L_3 \end{vmatrix} + z \begin{vmatrix} L_2 & L_3 \\ M_2 & M_3 \end{vmatrix} \right]$$

and two similar equations . . . . . (x)′.

The determinant of the transformation

$$(\xi'\eta'\zeta';\ \xi\eta\zeta) \text{ or } (xyz;\ x'y'z') = \frac{D}{\Delta} = \begin{vmatrix} L_1 & M_1 & N_1 \\ L_2 & M_2 & N_2 \\ L_3 & M_3 & N_3 \end{vmatrix}.$$

NOTE II.

The formula $D^2 = \Delta\Delta'$ of § 6 of this chapter, and the formula at the end of § 8 of chapter ii, are particular cases of the following:—

If the cosine of the angle between b_i and b_j—any two of

the n straight lines $b_1, b_2, b_3 \ldots, b_{n-1}, b_n$—be denoted by a_{ij} or a_{ji} (with a convention as to the positive direction of the straight lines), then

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdot & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdot & \cdot & \cdot & \cdot & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdot & \cdot & \cdot & \cdot & a_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & a_{n3} & \cdot & \cdot & \cdot & \cdot & a_{nn} \end{vmatrix} = 0; \text{ if } n \text{ is} > 3.$$

Take any point A_2 and draw a line through it in the direction of b_2; take any point A_3 on this line, and draw through it a line in the direction of b_3; on this take any point A_4 and draw a line through it in the direction of b_4 and so on, till we get to a line through a point A_{n-1} parallel to b_{n-1}.

Through A_2 draw a plane parallel to b_1 and b_n meeting this line in A_n; and through A_n draw a line (lying in this plane) in the direction of b_n, meeting a line through A_2 in the opposite direction to b_1 in the point A_1.

Then we have a closed polygon $A_1A_2A_3 \ldots A_{n-1}A_n$ whose sides are in the directions of $b_1, b_2, b_3, \ldots b_n$; let the length of its sides be respectively $x_1, x_2, x_3, \ldots x_n$. Projecting all the sides of this polygon on each side in turn we get the equations

$$\begin{aligned} x_1a_{11} + x_2a_{12} + x_3a_{13} + \ldots + x_na_{1n} &= 0. \\ x_1a_{21} + x_2a_{22} + x_3a_{23} + \ldots + x_na_{2n} &= 0. \\ x_1a_{31} + x_2a_{32} + x_3a_{33} + \ldots + x_na_{3n} &= 0. \\ \cdot \quad \cdot \quad \cdot \quad

CHAPTER VIII

CRYSTAL FORMS.

§ 1. Those crystals which are brought to self-coincidence by the operations of a group G (which must be one of the thirty-two groups discussed in chapters v and vi) and by no other operations, are said to form a *class*. This class may be denoted by the same symbol as the corresponding group. If we take those planes of the polyhedron representing a crystal (p. 36) which are equidistant from the centre O, and which are therefore parallel to faces having the same physical properties, and produce them till they meet again, the polyhedron* F so obtained is called a *form*. The feet of the perpendiculars from O on the faces of F are evidently a system of equivalent points; therefore the number of faces of any 'form' is *in general* equal to the number of operations in the corresponding group. As figures of various forms, both general and special, for each of the thirty-two crystal classes are given in nearly every textbook on the subject, we shall not reproduce them here, but shall content ourselves with proving a few general properties of such polyhedra.

§ 2. *If a crystal has a centre of symmetry, F has its faces parallel in pairs.*

For a plane is brought by inversion about a point O to coincidence with a parallel plane on the opposite side of and equidistant from O.

A symmetry-plane cannot meet a face of F except along one of its edges.

For if a face p of F meets a symmetry-plane in the line Σ; then p', the reflexion of p in the symmetry-plane, also passes through Σ and is a face of F; so that Σ is the intersection of two faces of F, and is therefore an edge of F or p.

* This polyhedron is not closed in the classes C_1, C_2, C_s, C_{2v}, C_4, C_{4v}, C_3, C_{3v}, C_6, C_{6v}, but is closed in the other twenty-one classes.

An n-al rotation-axis meets F in one of its solid angles, at which $2n$ or n edges of F intersect, according as the n-al axis does or does not lie in a symmetry-plane.

For if a face p of F meets the axis in A, then the positions into which p is brought by successive rotations through $\frac{2\pi}{n}$ about OA are also faces of F and pass through A; these n faces meet in n lines which are edges of F if there is no other symmetry-operation of the group which leaves A unmoved, i. e. if there is no symmetry-plane passing through OA. If however there is a symmetry-plane through OA, there are n other symmetry-operations which leave OA unmoved (cf. p. 49); hence there will be $2n$ faces of F passing through A, and therefore $2n$ edges of F passing through A. If $n = 2$, and the axis does not lie in a symmetry-plane, we have an exception; for two planes through a point do not meet in two lines but in one. Hence a 2-al rotation-axis not lying in a symmetry-plane does not pass through an angle of F, but meets an edge of F. Since a rotation through π about the axis brings this edge into self-coincidence, the axis must be perpendicular to the edge and pass through its middle point.

§ 3. Since any symmetry-operation of F brings each system of equivalent symmetry-elements of F into self-coincidence, therefore every face of F must be situated with regard to each equivalent set of symmetry-elements similarly to all the other faces.

Hence *each face must meet one and only one axis of each set of equivalent rotation-axes.*

For if any face met none of the set, then none of the faces could meet any axis of the set; let then each face meet λ axes of the set, which we will suppose contains a equivalent n-al rotation-axes. Let N be the number of the faces of F. Then since $2n$ or n faces meet in a point on each of the equivalent axes, according as the axes do or do not lie on symmetry-planes, therefore $a = \frac{\lambda N}{2n}$ or $\frac{\lambda N}{n}$ according as the axes do or do not lie on symmetry-planes. But (p. 50) $a = \frac{N}{2n}$ or $\frac{N}{n}$ according as the axes do or do not lie on symmetry-planes, and therefore in either case $\lambda = 1$.

§ 4. It may be easily verified (by considering the coordinates of systems of equivalent points or otherwise) that all the

groups in any one of the systems of § 2 of the previous chapter (p. 73) are subgroups of that group of the system which has the highest symmetry *; e. g.

$$C_3, C_{3i}, C_{3v}, D_3, D_{3d}, C_{3h}, D_{3h}, C_6, C_{6h}, C_{6v}, D_6,$$

are subgroups of D_{6h}. Now the number of operations of a subgroup is an aliquot part of the number of operations of the group itself (p. 47); hence to obtain a polyhedron which is brought to self-coincidence by the operations of any group G_1, we take a certain fraction of the faces of a polyhedron whose symmetry-operations form that group (G) of the system containing G_1 which has the highest symmetry, and extend them to form a (closed or unclosed) polyhedron again. The fraction is easily verified to be always $\frac{1}{2}$, $\frac{1}{4}$, or $\frac{1}{8}$.

Polyhedra whose symmetry-operations form a group such as G are called *holohedral* †; those whose symmetry-operations form a group such as G_1 are called *merohedral* ‡ in general; or 'hemihedral,' 'tetartohedral,' or 'ogdohedral' § according as G_1 contains $\frac{1}{2}$, $\frac{1}{4}$, or $\frac{1}{8}$ of the operations of G.

Similarly G is called a 'holohedral group,' G_1 a 'merohedral' (or 'hemihedral,' &c.). The words 'holohedry,' 'merohedry,' &c., are often used for 'holohedral groups,' 'merohedral groups,' &c.

The groups T_d, C_{6v}, C_{4v}, C_{3v} ||, C_{2v}, C_2 are often called '*hemimorphic* ¶.' Their polyhedra possess those faces of the corresponding holohedral polyhedra which meet *one* end of the 3-al axes, or 6-al, 4-al, 3-al, 2-al axis respectively.

The groups T_h, C_{6h}, C_{4h}, C_{3h} which have a symmetry-plane perpendicular to a symmetry-axis, but which are not holohedral, are sometimes called '*paramorphic*.'

The groups O, D_6, D_4, D_3 ||, D_2, (or Q) are sometimes called '*enantiomorphous*.' They possess no operations of the second sort, and the corresponding polyhedra are not superposable on their own mirror-images in any plane. If p, p' are any two parallel planes equidistant from and on opposite sides of O; and if F, F' are the polyhedra formed by planes equivalent to p, p' respectively in any one of these five classes; then F, F' are enantiomorphous to one another, and

* This is true if the first subdivision of the hexagonal system is treated as a separate system, for C_3, C_{3i}, C_{3v}, D_3 are subgroups of D_{3d}.

† From ὅλος, whole; ἕδρα, a face.

‡ μέρος, a part; ἕδρα, a face.

§ This only occurs in the case of C_3, and then only if the rhombohedral system is treated as a subdivision of the hexagonal.

|| If the rhombohedral system is considered distinct from the hexagonal.

¶ Or 'Antimorphic.'

the faces of F and F' taken together make up the faces of the corresponding holohedral polyhedron.

§ 5. So far we have considered any face p of F to have a general position, and in that case each group has its own characteristic polyhedron (or 'form'), which has as many faces as the group has operations. If, however, p has a special position this is no longer the case. Thus, for example, if the plane p is perpendicular to a 4-al axis of O_h or O, or to a 2-al axis of T_d, T_h, or T; the form for all the groups O_h, O, T_d, T_h, and T is a cube. If p is perpendicular to a 3-al axis of any of these groups, the form is a regular octahedron for O_h, O, and T_h, and a regular tetrahedron for T_d and T.

These special forms are extremely common in the crystals found in nature, and the non-existence of general forms often makes it very difficult to determine the true symmetry of any given crystal.

CHAPTER IX

THE CRYSTALLOGRAPHIC AXES AND AXIAL RATIOS.

§ 1. We shall in this chapter show how to choose most conveniently the position of the crystallographic axes of any crystal, and the magnitude of the axial ratios (§§ 2 to 8); and how to find the indices of all faces belonging to the same form as any face whose indices are given.

§ 2. For the triclinic system we choose lines parallel to *any* three edges * as crystal axes, and *any* face as 'parametral face.' The axial ratios are perfectly general.

§ 3. In the monoclinic system the symmetry-axis (or the line perpendicular to the symmetry-plane in the case of C_s) is always parallel to a possible edge, and in the plane perpendicular to it there is an indefinite number of lines parallel to possible edges (chap. iii, §§ 17 and 18).

Take then the symmetry-axis (the line perpendicular to the symmetry-plane) as the crystallographic axis OC † (cf. chap. ii, § 2, p. 9), and any two lines parallel to possible edges and perpendicular to OC (not in general at right angles to one another) as the axes OA and OB. Take *any* face as parametral face.

§ 4. In the classes D_2, D_{2h} we take the three symmetry-axes as crystallographic axes, which is possible, for they are parallel to possible edges. For C_{2v} we take the symmetry-axis as the axis OC, and the intersections of the plane perpendicular to this axis (which is parallel to a possible face) with the symmetry-planes (which are also parallel to possible faces) as the other two crystallographic axes.

We take *any* face as parametral face.

* It will be remembered that the crystallographic axes must of necessity be parallel to crystal edges (p. 9).

† In books on Descriptive Crystallography it is usual to take OB parallel to the symmetry-axis, instead of OC as in books on Theoretical Crystallography.

§ 5. For the tetragonal system we take the 4-al axis of the first or second sort as the crystallographic axis OC.

For C_4', C_4 and C_{4h} we take a line parallel to any edge perpendicular to OC as the crystallographic axis OA; for C_{4v} we take a line perpendicular to OC and lying in a symmetry-plane; and for D_{2d}, D_4, and D_{4h} we take a 2-al rotation-axis. In each case we take the line with which OA would be made to coincide by a rotation through $\frac{\pi}{2}$ about OC as the axis OB.

Now there is always a possible face making equal intercepts on OA, OB (p. 42). Take such a face as parametral face; then the axial ratios may be written $a:a:c$ $(a = b)$.

§ 6. The following method may be applied in the case of the rhombohedral system. Choose as the axis OC, for C_3 and C_{3i} a line* parallel to any edge, for C_{3v} and D_{3d} a line* parallel to an edge and lying in a symmetry-plane, for D_3 (and D_{3d}) a line* parallel to an edge and perpendicular to a 2-al axis. As the axes OA, OB choose the positions into which OC would be brought by successive rotations through $\frac{2\pi}{3}$ about the 3-al symmetry-axis.

Then assuming that there is a possible face perpendicular to the symmetry-axis†, we take this for parametral face, and may then write our axial ratios

$$a:a:a\,(a = b = c).$$

§ 7. For the hexagonal system as a whole, including the rhombohedral system as a subdivision, we take the 3-al or 6-al rotation-axis as the axis OC. For C_3, C_{3i}, C_{3h}, C_6, C_{6h} we take any line perpendicular to OC and parallel to a possible edge as the axis OA‡; for C_{3v}, C_{6v} we take a line perpendicular to OC and lying in a symmetry-plane; for D_3, D_{3d}, D_{3h}, D_6, D_{6h} we take a 2-al rotation-axis. We now take *two* more crystallographic axes OB, OD (thus having four crystallographic axes in all) such that OA would be brought into the positions OB, OD by successive rotations through $\frac{2\pi}{3}$ about OC.

* Not perpendicular to, or coinciding with the symmetry-axis.

† This can be *proved* for C_{3v}, D_3 and D_{3d} by the aid of §§ 17 and 18 of chap. iii.

‡ It is assumed (not proved) that there is such a line in the case of C_3 and C_{3i}.

Let any face meet the axes OA, OB, OD in A', B', D' (Fig. 59).

Then
$$\frac{\sin\left(OB'D' - \frac{\pi}{3}\right)}{\sin OB'D'} = \frac{\sin OA'B'}{\sin OB'A'} = \frac{B'O}{OA'},$$
$$\frac{\sin\left(OB'D' + \frac{\pi}{3}\right)}{\sin OB'D'} = \frac{\sin OD'B'}{\sin OB'D'} = \frac{B'O}{OD'}.$$

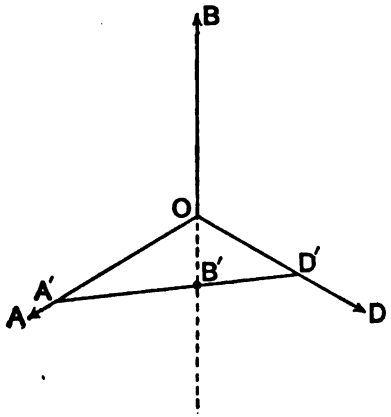

Fig. 59.

Therefore by addition we have
$$\frac{1}{OA'} + \frac{1}{OB'} + \frac{1}{OD'} = 0.$$

Now there is a crystal face making equal intercepts on OA and OD; therefore $OA' : OD'$ is always rational.

But
$$\frac{1}{OA'} + \frac{1}{OB'} + \frac{1}{OD'} = 0,$$
and hence $OA' : OB' : OD'$ are rational ratios. Therefore the intercepts of any face on the four axes, OA, OB, OD, OC, are proportional to $\frac{a}{h}, \frac{a}{k}, \frac{a}{i}, \frac{c}{l}$ where a and c are some fixed quantities and h, k, i, l are integers. The ratios $a:a:a:c$ are called the axial ratios, and h, k, i, l the indices of the face.

Since $\frac{1}{OA'} + \frac{1}{OB'} + \frac{1}{OD'} = 0$, $h+k+i=0$.

§ 8. In the regular system we take the three 4-al rotation-axes as crystallographic axes for O, O_h; and for T, T_h, T_d we take the three 2-al rotation-axes. In either case there is a possible face perpendicular to each of the 3-al axes*. Taking one of these as parametral face, we may write the axial ratios
$$a:a:a \quad (a=b=c).$$

§ 9. We thus see that we are able to choose as crystallographic axes lines similar to those which we have chosen in chap. vii as axes of coordinates.

* This is assumed for T and T_h; it may be proved for T_d, O, and O_h.

	Symbol of the crystal class *.	Name of the class adopted in the rest of this book *.	Number and name used by Professor H. A. Miers †.	Number and name used by Professor E. S. Dana ‡§.	Other names in general use (if any).	Number of faces in the general form.	Mineral typical of the class ‖.	Artificial crystal belonging to the class ‖.
Triclinic	C_1 C_i	Hemihedry Holohedry	i. Asymmetric ii. Centrosymmetric	32. Asymmetric 31. Normal	Hemipinakoidal, Pedial Pinakoidal	1 2	Axinite	$CaS_2O_3 . 6H_2O$
Monoclinic	C_s	Hemihedry						

Let any crystal face whose indices are h, k, l meet these axes in A', B', C', and the perpendicular from O on the face in M' (Fig. 10, p. 17); and let the Fedorow coordinates of the point M' be $\xi' \eta' \zeta'$.

Then $$h:k:l = \frac{a}{OA'} : \frac{b}{OB'} : \frac{c}{OC'}$$

$$= a\cos M'OA' : b\cos M'OB' : c\cos M'OC' = a\xi' : b\eta' : c\zeta'.$$

Now an operation which brings the face (hkl) to coincide with some other face p brings M' to coincide with the perpendicular from O on p. Hence to find the indices of all faces equivalent to (belonging to the same form as) the face (hkl), it is sufficient to know the Fedorow coordinates of all points equivalent to the point $(\xi' \eta' \zeta')$. These are given in chapter vii. Using the systems of coordinates there given and the axial ratios obtained in this chapter, we find at once that in any class *the system of indices of faces equivalent to* (hkl) *is of exactly the same type as the system of Fedorow coordinates of points equivalent to* $(\xi \eta \zeta)$.

Thus as an example take the class C_4. The Fedorow coordinates of a system of equivalent points are (p. 85)

$$\xi\eta\zeta,\ {}^{-}\eta\xi\zeta,\ {}^{-}\xi{}^{-}\eta\zeta,\ \eta{}^{-}\xi\zeta.$$

Therefore the system of faces equivalent to (hkl) is, since $b = a$, $(a\xi'\ a\eta'\ c\zeta')$, $(a{}^{-}\eta'\ a\xi'\ c\zeta')$, $(a{}^{-}\xi'\ a{}^{-}\eta'\ c\zeta')$, $(a\eta'\ a{}^{-}\xi'\ c\zeta')$ or (hkl), $(\bar{k}hl)$, $(\bar{h}\bar{k}l)$, $(k\bar{h}l)$.

The symbol $\{hkl\}$ is often used to denote the form to which the face (hkl) belongs.

Again, we have another proof of the fact that in the hexagonal system $h+k+i = 0$ (p. 96); for

$$\xi_1' + \xi_2' + \xi_3' = 0 \text{ and } h:k:i = \xi_1' : \xi_2' : \xi_3'.$$

§ 10. We sum up the crystal classes in the appended table. The first three columns contain the names and symbols adopted in the rest of this book; they are based on Schoenflies' notation.

The symmetry of some of the crystals in the last two columns of the table is not quite certain.

CHAPTER X

THE DEPENDENCE OF PHYSICAL PROPERTIES OF CRYSTALS ON SYMMETRY.

§ 1. Physical phenomena of solids are of three kinds, scalar, vectorial, and tensorial. Scalar phenomena are those which can be expressed in terms of magnitude, but not of direction; well-known examples are temperature, density, electrical potential, and sublimation vapour pressure.

Vectorial phenomena are those which can be expressed in terms of magnitude and direction, and can therefore be geometrically represented by a finite straight line of known length and direction; examples are transmission of light, thermal and electrical conductivity, &c.

Tensorial phenomena are those which can be expressed in terms of magnitude and two opposed directions, and can be geometrically represented by two finite straight lines equal, and in opposite directions; such phenomena are strain and stress.

§ 2. The usual problems we have to consider in crystal-physics consist in determining the dependence of one phenomenon (A) on another (B) when the material in which the phenomena are manifested is crystalline; thus e. g. it may be required to find the electrical current which passes along a thin rod (of known length and cross section) which is cut from a crystal *in any given direction*, when the difference of potential at the ends of the rod is given.

If both A and B are scalar (e. g. if we are considering the influence of temperature on density) we can draw no distinction between amorphous and crystalline media.

§ 3. If B is scalar, and A vectorial (e. g. if we are considering the property of pyroelectricity, i. e. the appearance of electrical

polarity on crystals with change of temperature), a distinction between amorphous and crystalline matter at once arises.

Let us take any three orthogonal reference-axes OX, OY, OZ. Now A can be represented by a straight line through O; let its length be ρ and its direction-cosines l, m, n; then A may be resolved into three components of magnitudes, ρl, ρm, ρn, along OX, OY, and OZ: B, being scalar, is represented by a magnitude θ.

Then we have relations of the form *

$$\left.\begin{aligned} \rho l &= l_1 \theta \\ \rho m &= l_2 \theta \\ \rho n &= l_3 \theta \end{aligned}\right\},$$

where l_1, l_2, l_3 are constants depending on the nature of the medium and the positions of the reference-axes.

Consider now the ellipsoid $\frac{x^2}{l_1^2}+\frac{y^2}{l_2^2}+\frac{z^2}{l_3^2} = 3\theta^2$; the length of that radius vector whose direction cosines are l, m, n is

$$\sqrt{3\theta^2 \div \left(\frac{l^2}{l_1^2}+\frac{m^2}{l_2^2}+\frac{n^2}{l_3^2}\right)} = \rho.$$

Hence the relation between the direction and magnitude of the straight line, which represents A when θ is given, is geometrically represented by the relation between the direction and magnitude of the radius vector from the centre of a certain ellipsoid. The dimensions and orientation of this ellipsoid must be independent of our reference-axes, for it represents a physical property of the crystalline material.

§ 4. If A and B are both vectorial (e. g. if we are considering the dependence of electrical current on electromotive force), let the lengths and direction-cosines of the straight lines representing A and B be respectively ρ, l, m, n; R, L, M, N. The components of these lines along the reference-axes are $\rho l, \rho m, \rho n$; RL, RM, RN. Now in a very large number of important cases it is found that $\rho l, \rho m, \rho n$ can be expressed *linearly* in terms of RL, RM, RN; at least to a first approximation.

We have then equations of the form

$$\left.\begin{aligned} \rho l &= a_{11} RL + a_{12} RM + a_1 RN \\ \rho m &= a_{21} RL + a_{22} RM + a_{23} RN \\ \rho n &= a_{31} RL + a_{32} RM + a_{33} RN \end{aligned}\right\} \quad \ldots \quad \text{(i)}.$$

* These linear relations are in most cases found to hold good, at any rate as a first approximation.

Now again in some cases we can prove by thermodynamical considerations *, or failing these experimentally, that

$$a_{12} = a_{21},\ a_{31} = a_{13},\ a_{23} = a_{32}.$$

If that is so, consider the ellipsoid

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{23}yz + 2a_{31}zx + 2a_{12}xy = 1 \quad . \quad . \quad \text{(ii)}.$$

Let r be the length of that radius vector whose direction-cosines are L, M, N; then the tangent plane at the extremity of this radius vector is

$$x(a_{11}L + a_{12}M + a_{31}N) + y(a_{12}L + a_{22}M + a_{23}N) + z(a_{31}L + a_{23}M + a_{33}N) = \frac{1}{r},$$

or

$$xl + ym + nz = \frac{R}{\rho}\cdot\frac{1}{r}.$$

The perpendicular (p) from the centre on this plane has the direction-cosines l, m, n and is of length $\frac{R}{\rho}\cdot\frac{1}{r}$; so that the ellipsoid (ii) gives us the following method of finding geometrically the relation between the lines representing A and B. Take the radius vector (r) whose direction-cosines are those of the line (of length R) representing B, and draw a tangent plane at its extremity; then the straight line which represents A is parallel to the perpendicular (p) from the centre on this tangent plane, and its length ρ is given by the relation $\rho pr = R$.

It should be noticed that if B is of such a magnitude that it is represented (in magnitude and direction) by a radius vector of the ellipsoid (ii), A is represented by the corresponding vector of the polar reciprocal of (ii) with respect to a concentric sphere of unit radius.

As before the dimensions and orientation of the ellipsoid (ii) cannot depend on the position of the axes of reference.

Even if we do not know that in equations (i)

$$a_{12} = a_{21},\ a_{31} = a_{13},\ a_{23} = a_{32}$$

we can represent the physical property under discussion *partially* by means of an ellipsoid. For squaring and adding equations (i) we have

$$\rho^2 = R^2(aL^2 + bM^2 + cN^2 + 2fMN + 2gNL + 2hLM),$$

* e.g. when we discuss the dependence of the electrical moment of a dielectric on the electrostatic field.

where $a = a_{11}^2 + a_{21}^2 + a_{31}^2$, $f = a_{12} \cdot a_{13} + a_{22} \cdot a_{23} + a_{32} \cdot a_{33}$, &c., since $l^2 + m^2 + n^2 = 1$.

Hence $\frac{R}{\rho}$ = the length of the radius vector of the ellipsoid

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + hxy = 1 \quad . \quad . \quad . \quad . \quad \text{(iii)},$$

drawn from the centre parallel to the straight line representing B.

Since this ellipsoid only represents the connexion between the magnitude and direction of B with the *magnitude* (not the direction) of A, the geometrical representation of the property under discussion is not so complete as in the case when $a_{12} = a_{21}$, $a_{31} = a_{13}$, $a_{23} = a_{32}$.

The dimensions and orientation of the ellipsoid (iii) do not depend on the choice of axes.

§ 5. We mentioned before * that all physical properties of a given crystal must share its symmetry, and that a crystal cannot have a higher symmetry than the least symmetrical of its physical properties. In general the physical properties are of a higher order of symmetry than the geometrical (the occurrence of 'forms'), but the true symmetry of the latter is not always easy to determine, partly owing to the frequent non-occurrence of any but special forms on crystals found in nature, and partly owing to the simulation by crystals of higher symmetry than they really possess by means of twinning, &c. (see next chapter).

In those cases where the mutual dependence of two phenomena can be represented by an ellipsoid, the ellipsoid must possess the symmetry of the crystal. Now the elements of symmetry possessed by *every* ellipsoid are three mutually perpendicular 2-al rotation-axes, a centre of symmetry, and three planes of symmetry perpendicular to the axes (in fact the symmetry-elements of the group D_{2h}). An ellipsoid cannot possess a 3-al, 4-al, &c., axis unless it is a figure of revolution about that axis; it cannot possess more than one axis other than 2-al axes unless it is a sphere.

§ 6. We can therefore divide crystals into five classes by means of their behaviour towards those physical properties which can be represented by an ellipsoid. Such a property is the propagation of light (for it can be represented by the Fresnel ellipsoid), and we shall use that as our illustration.

(1) In the regular system the ellipsoid must be a sphere;

* p. 36.

all crystals of the regular system are 'singly refracting,' or 'isotropic as regards light *.'

(2) In the hexagonal, tetragonal, and rhombohedral systems the ellipsoid must be a figure of revolution whose axis coincides with the 6-al, 4-al, or 3-al symmetry-axis; all crystals of these systems are uniaxial †.

(3) In the orthorhombic system the axes of the ellipsoid must coincide with those directions which we took as axes of reference in chapters vii and ix (pp. 84 and 94); all crystals of the orthorhombic systems are biaxial †, and the directions of the principal axes of the Fresnel ellipsoid are the same for every wave length of light.

(4) In the monoclinic system *one* of the axes of the ellipsoid must coincide with the axis of symmetry or be perpendicular to the symmetry-plane; all crystals of the monoclinic system are biaxial, and the direction of *one* of the principal axes of the Fresnel ellipsoid is the same for every wave length of light, but the directions of the other two axes in general vary with the wave length.

(5) In the triclinic system there is no position with which an axis of the ellipsoid must necessarily coincide; all crystals of this class are biaxial, and the directions of all the axes of the Fresnel ellipsoid vary with the wave length.

§ 7. Consider now a more general case. Suppose that the phenomena A, B are represented by straight lines Oa, Ob whose lengths are ρ, R, and direction-cosines (referred to orthogonal axes OX, OY, OZ) l, m, n; L, M, N; and suppose that we have relations of the form

$$\left.\begin{aligned} \rho l &= f_1(RL, RM, RN) \\ \rho m &= f_2(RL, RM, RN) \\ \rho n &= f_3(RL, RM, RN) \end{aligned}\right\} \quad \ldots\ldots\ldots \text{(i)},$$

whereby ρl, ρm, ρn are expressed algebraically in terms of RL, RM, RN and certain constants (corresponding to the a^s of equations (i) in § 4) which depend only on the nature of the crystalline medium and the choice of reference-axes.

Now in a medium of given symmetry we can deduce certain relations between these constants as follows.

Let any symmetry-operation of the medium bring Oa, Ob, OX, OY, OZ into the positions Oa', Ob', OX', OY', OZ' respec-

* Sometimes called simply 'isotropic,' but it seems better to reserve this term for those solids whose behaviour towards *all* physical phenomena can be represented by a sphere; i.e. glassy or amorphous bodies.

† Or uniaxal, biaxal; both methods of spelling are in use.

tively; and let the direction-cosines of Oa', Ob' referred to the axes OX, OY, OZ be l', m', n'; L', M', N'.

Then since, if B is represented by Ob, A is represented by Oa; therefore by the symmetry of the crystal if B is represented by Ob', A must be represented by Oa'; and hence we must have

$$\left.\begin{aligned} \rho l' &= f_1(RL', RM', RN') \\ \rho m' &= f_2(RL', RM', RN') \\ \rho n' &= f_3(RL', RM', RN') \end{aligned}\right\} \quad \ldots\ldots \text{(ii)}.$$

Now by the formulae of chap. vii, §§ 8 and 9 (pp. 82, 83), we can express the direction-cosines of Oa', Ob' referred to OX, OY, OZ (i.e. l', m', n'; L', M', N') in terms of the direction-cosines of Oa', Ob' referred to OX', OY', OZ', which are evidently the same as the direction-cosines of Oa, Ob referred to OX, OY, OZ (i.e. l, m, n; L, M, N); we have in fact relations of the form

$$\left.\begin{aligned} l &= C_{11}l' + C_{12}m' + C_{13}n' \\ m &= C_{21}l' + C_{22}m' + C_{23}n' \\ n &= C_{31}l' + C_{32}m' + C_{33}n' \\ L &= C_{11}L' + C_{12}M' + C_{13}N' \\ M &= C_{21}L' + C_{22}M' + C_{23}N' \\ N &= C_{31}L' + C_{32}M' + C_{33}N' \end{aligned}\right\} \quad \ldots\ldots \text{(iii)},$$

where every C is known.

Substituting in (i) from (iii) we have

$$C_{11}\rho l' + C_{12}\rho m' + C_{13}\rho n' = f_1(C_{11}RL' + C_{12}RM' + C_{13}RN', C_{21}RL' + C_{22}RM' + C_{23}RN', C_{31}RL' + C_{32}RM' + C_{33}RN') = \phi_1(RL', RM', RN') \text{ say}$$

and two similar equations.

Solving these we have (remembering that $\begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix} = \pm 1$)

$$\rho l' = \pm \begin{vmatrix} \phi_1(RL', RM', RN') & C_{12} & C_{13} \\ \phi_2(RL', RM', RN') & C_{22} & C_{23} \\ \phi_3(RL', RM', RN') & C_{32} & C_{33} \end{vmatrix}$$

and two similar equations . . . . . (iv).

These equations (iv) must be identical with (ii); and therefore

$$f_1(RL', RM', RN') \equiv \pm \begin{vmatrix} \phi_1(RL', RM', RN') & C_{12} & C_{13} \\ \phi_2(RL', RM', RN') & C_{22} & C_{23} \\ \phi_3(RL', RM', RN') & C_{32} & C_{33} \end{vmatrix}$$

and two similar identities.

Expanding $f_1, f_2, f_3, \phi_1, \phi_2, \phi_3$ in powers and products of

powers of RL', RM', RN', and equating coefficients on the two sides of these three equations, we get some of the relations between the constants which we required; we obtain the remainder by using other symmetry-operations of the medium. Not all the relations so obtained are necessarily independent.

§ 8. When either A or B is a tensorial phenomenon the mathematical relations are more complicated, since in this case the phenomenon cannot well be represented by the resultant of three components. For these and other developments of this branch of the subject we must refer the reader to W. Voigt's "Elemente der Krystallphysik," Th. Liebisch's "Physikalische Krystallographie," &c. For instance, it is there shown that the elasticity of a triclinic crystal can be represented by a surface involving 21 constants. A property of considerable importance in theories of crystal-structure, and one which may have some connexion with elasticity, is that of cleavage, or the tendency of crystals to break in such a way that the surface of fracture is a plane. This plane is always a possible face of the crystal, and the crystal may with equal ease be broken so that any other face of the same form is the fracture-surface; in other words the cleavage shares the symmetry of the crystal. Very often two (or more) faces belonging to different forms can be obtained by fracture in this way, but not in general with equal ease. Forms obtained by cleavage may or may not occur as forms found on a crystal grown under the ordinary conditions. The mathematical problems connected with cleavage (e.g. its connexion with elasticity) still await solution.

CHAPTER XI

ON THE GROWTH OF CRYSTALS.

§ 1. In this chapter we propose to give a very brief account of the phenomena observed in processes connected with the growth of crystals. As is well known, a material crystallizes out from a supersaturated solution, from a liquid cooled to its freezing-point, or from a supersaturated vapour *.

We shall first consider a crystal which has grown without any extraneous disturbance, and shall assume that the polyhedron formed by all faces of such a crystal has the full symmetry of the ideal crystal of p. 36; in other words we shall assume that all faces whose physical properties are the same are equidistant from a fixed point which we have called the centre of the crystal. The faces of the crystal do not all necessarily belong to the same 'form' (i. e. are not all equivalent to one another), but if one face of a form appears all the rest must appear also and be equidistant from the centre. Faces belonging to the same form must be alike in all their physical properties; but faces belonging to different forms differ in hardness, lustre, &c., and are not in general at the same distance from the centre. A crystal may possess faces belonging only to one form, if that form is closed; otherwise the faces must belong to two or more forms †. Even if the symmetry allows of all the faces belonging to a single closed form, very often the crystal possesses faces belonging to several different forms.

§ 2. Among other properties which are different for faces belonging to different forms must be the tension (due to capillary action) of the surfaces separating a crystal from the liquid or vapour in which it is growing. This tension may

* This process is called 'sublimation'; good examples are camphor, phosphorus, and ammonium chloride.

† For example, crystals whose symmetry is that of C_1 must possess faces belonging to 4 forms, crystals whose symmetry is that of C_2 must possess faces belonging to 3 forms, crystals whose symmetry is that of C_{∞} must possess faces belonging to 2 forms, &c.

be measured by a constant for each face which we shall call the capillarity constant. A crystal must always grow (if undisturbed) so that its surface energy is a minimum for a given volume of solid matter crystallized out; hence if $s_1, s_2, s_3, \ldots$ are the areas of the various faces, $k_1, k_2, k_3, \ldots$ the corresponding capillarity constants, then $E \equiv k_1 s_1 + k_2 s_2 + k_3 s_3 + \ldots$ must be a minimum for a given volume of the crystal*. There is an interesting connexion between the capillarity constants of the various faces and their distance from the centre of the crystal†, which may be expressed in the following theorem:—

The perpendiculars on the faces of a crystal from a certain point within it are proportional to the capillarity constants of the faces.

Let I be any point within the crystal and let $n_1, n_2, n_3, n_4 \ldots$ be the perpendiculars from I on the faces $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \ldots$ whose areas are $s_1, s_2, s_3, s_4, \ldots$, and whose capillarity constants are $k_1, k_2, k_3, k_4, \ldots$. Let V be the volume of the crystal, and let $P_1, P_2, P_3, P_4, \ldots$ be the planes through I parallel to $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \ldots$. Now there are certain geometrical relations between the quantities $s_1, s_2, s_3, s_4, \ldots$ which depend on the fact that they are the areas of polyhedron faces whose normals have fixed directions.

For let A be any point; $n_{1a}, n_{2a}, n_{3a}, n_{4a}, \ldots$ the perpendiculars from A on $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \ldots$; $a_1, a_2, a_3, a_4 \ldots$ ‡ the perpendiculars from A on $P_1, P_2, P_3, P_4 \ldots$.

Then $n_1 - n_{1a} = a_1$, $n_2 - n_{2a} = a_2$, &c.

Since $V = \frac{1}{3} \Sigma n s = \frac{1}{3} \Sigma n_a s$,

$$\therefore \quad \Sigma (n - n_a) s = 0,$$

and $$\therefore \quad \Sigma a s = 0.$$

$a_1, a_2, a_3, a_4, \ldots$ are not independent, for the position of A is completely defined by the perpendiculars from it to the *three* planes P_1, P_2, P_3. We must have linear relations of the form $a_4 = p_4 a_1 + q_4 a_2 + r_4 a_3$, $a_5 = p_5 a_1 + q_5 a_2 + r_5 a_3$, &c.

Now take other points $B, C, D, \ldots$ such as A; then we shall have an indefinite number of similar relations

$$\Sigma b s = \Sigma c s = \Sigma d s = \ldots = 0.$$

* Of course if two or more of the faces belong to the same form the corresponding areas and capillarity constants must be equal.

† If the growth of the crystal is undisturbed.

‡ a_1 is reckoned positive or negative according as A is on the same or on the opposite side of P_1 as σ_1, and so for $a_2, a_3, a_4 \ldots$.

These are, however, only equivalent to *three* independent relations.

For choose quantities λ, μ, ν to satisfy the equations

$$d_1 = \lambda a_1 + \mu b_1 + \nu c_1;\ d_2 = \lambda a_2 + \mu b_2 + \nu c_2;\ d_3 = \lambda a_3 + \mu b_3 + \nu c_3.$$

Then $d_4 = p_4 d_1 + q_4 d_2 + r_4 d_3 = p_4(\lambda a_1 + \mu b_1 + \nu c_1)$
$+ q_4(\lambda a_2 + \mu b_2 + \nu c_2) + r_4(\lambda a_3 + \mu b_3 + \nu c_3)$
$= \lambda(p_4 a_1 + q_4 a_2 + r_4 a_3) + \mu(p_4 b_1 + q_4 b_2 + r_4 b_3)$
$+ \nu(p_4 c_1 + q_4 c_2 + r_4 c_3) = \lambda a_4 + \mu b_4 + \nu c_4.$

Similarly

$$d_5 = \lambda a_5 + \mu b_5 + \nu c_5, \text{ \&c., and } \Sigma d s \equiv \lambda \Sigma a s + \mu \Sigma b s + \nu \Sigma c s.$$

Suppose now the crystal to receive a small deformation, such that the normals to its faces are fixed in direction; then the volume of matter added on a face σ lies between $s dn$ and $(s + ds) dn$ and therefore $= s dn$, neglecting small quantities of the second order.

Hence $dV = \Sigma s dn$.

Again $dV = d\{\frac{1}{3}\Sigma n s\} = \frac{1}{3}\Sigma\{n ds + s dn\} = \frac{1}{3}[\Sigma\{n ds\} + dV]$

$\therefore\ \Sigma n ds = 2 dV.$

Now if the surface energy $E \equiv \Sigma k s$ is a minimum when V is constant, we must have $\Sigma n ds = \Sigma k ds = 0$; and also we have $\Sigma a ds = \Sigma b ds = \Sigma c ds = 0$.

Since the quantities ds_1, ds_2, ds_3, ds_4, ... are connected by these five equations, and by these only, we have by the ordinary rules of the differential calculus

$$\begin{aligned} n_1 - a a_1 - \beta b_1 - \gamma c_1 &= \rho k_1 \\ n_2 - a a_2 - \beta b_2 - \gamma c_2 &= \rho k_2 \\ n_3 - a a_3 - \beta b_3 - \gamma c_3 &= \rho k_3 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{aligned}$$

where a, β, γ, ρ are (unknown) constants.

Now choose a point I' whose distances from P_1, P_2, P_3 are respectively $a a_1 + \beta b_1 + \gamma c_1$, $a a_2 + \beta b_2 + \gamma c_2$, $a a_3 + \beta b_3 + \gamma c_3$; then its distance from P_4 is

$$\begin{aligned} &p_4(a a_1 + \beta b_1 + \gamma c_1) + q_4(a a_2 + \beta b_2 + \gamma c_2) + r_4(a a_3 + \beta b_3 + \gamma c_3) \\ &= a(p_4 a_1 + q_4 a_2 + r_4 a_3) + \beta(p_4 b_1 + q_4 b_2 + r_4 b_3) \\ &\quad + \gamma(p_4 c_1 + q_4 c_2 + r_4 c_3) = a a_4 + \beta b_4 + \gamma c_4. \end{aligned}$$

Similarly its distance from P_5 is $a a_5 + \beta b_5 + \gamma c_5$, &c.

Let its distances from σ_1, σ_2, σ_3, σ_4 ... be n_1', n_2', n_3', n_4' ..., then

$$n_1' = n_1 - a a_1 - \beta b_1 - \gamma c_1,\ n_2' = n_2 - a a_2 - \beta b_2 - \gamma c_2,$$
$$n_3' = n_3 - a a_3 - \beta b_3 - \gamma c_3,\ n_4' = n_4 - a a_4 - \beta b_4 - \gamma c_4, \text{ \&c.}$$

$$\therefore\ n_1' : n_2' : n_3' : n_4' : \ldots = k_1 : k_2 : k_3 : k_4 : \ldots *.$$

* In some cases it is easier (instead of using this geometrical property to

Since k is the same for all faces belonging to the same form, I' must be equidistant from all faces of the same form.

It may prove impossible to find a point whose distances from the faces $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \ldots$, are in the ratios $k_1 : k_2 : k_3 : k_4 : \ldots$; or even if it is possible, it may be found that E is a maximum, not a minimum. That means that a crystal of the material considered cannot grow with this particular combination of faces, but must grow with some other combination.

The capillarity constants vary with the medium with which the crystal is in contact; hence it is easy to account for the observed fact that the relative predominance of different faces on crystals grown from a solution often alters if a foreign substance is added to the solution.

§ 3. Suppose that when the crystal is very small (when it is just beginning to grow) its position is J; and let the thicknesses of the crusts of the solid (measured parallel to the normals) deposited on the faces $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \ldots$ in a small period of time be in the ratios $h_1 : h_2 : h_3 : h_4 : \ldots$.

Now, assuming that the rates of growth parallel to the normals of the different faces are always in constant ratios *, it is evident that the faces of the crystal in *all* stages of its growth are at distances from J which are in the ratios of $h_1 : h_2 : h_3 : h_4 : \ldots$.

From considerations of symmetry the rates of growth on all faces belonging to the same form must be equal, and therefore J must be always equidistant from faces of the same form.

In the case of crystals belonging to twenty-two of the thirty-two classes there is only one point which is always equidistant from all faces of the same form; the points J and I' (§ 2) must coincide at the 'centre' of the crystal, and we must have $h_1 : h_2 : h_3 : h_4 : \ldots = k_1 : k_2 : k_3 : k_4 : \ldots$. In the case of crystals belonging to the classes C_1, C_s, C_2, C_{2v}, C_3, C_{3v}, C_4, C_{4v}, C_6, C_{6v} there are an indefinite number of points equidistant from all faces of the same form †; in these classes

determine the shape of the crystal, assuming the capillarity constants known) to write down E and V in terms of suitably chosen independent variables; and to find what values of these variables make E a minimum and V constant, by the ordinary methods of the differential calculus.

* Just as we assumed that the capillarity constants depend only on the directions and not on the sizes of the faces.

† In the case of C_1 *any* point, in the case of C_s any point on the symmetry-plane, and in the case of the other eight classes any point on the symmetry-axis is equidistant from all faces of the same form. The position of the

J does not necessarily coincide with I', and we have no proof that $h_1 : h_2 : h_3 : h_4 : \ldots = k_1 : k_2 : k_3 : k_4 : \ldots$.

G. Wulff* has attempted to compare the capillarity constants of different faces belonging to monoclinic crystals of ferrous ammonium sulphate by measuring the rates of growth on these faces. Elaborate precautions were necessary in order to eliminate the effects of disturbing forces during growth.

§ 4. Such crystals as we have been considering are those ideal crystals which may be supposed grown indefinitely slowly at constant temperature. Crystals found in nature have very rarely been grown under circumstances which even approximate to those necessary for ideal growth.

Take for instance a crystal grown from solution. The latter is usually considerably supersaturated, there is consequently a rapid deposition of solid on the faces of the crystal; the portion of liquid nearest the crystal becomes less supersaturated and consequently less dense than the portions further away; and before diffusion has time to make good the difference, convection currents are set up in the liquid, which exercise a disturbing influence on the growth of the crystal. Moreover, very often a crystal in growing comes in contact with other solid bodies and has not free room to develop.

§ 5. We have based our conclusions in previous chapters on the 'law of rational indices.'

Sometimes however, at least during the process of growth, faces make their appearance which are inclined at a small but measurable angle (a few minutes) to the position which would give them simple rational indices. Thus Professor Miers has shown that growing crystals of potash alum belonging to the regular system, which at first glance appear to have only the eight faces belonging to the form {111}†, have in reality twenty-four faces. These are arranged in threes making small angles with the positions which would be occupied by the faces of {111}; and they make triangular pyramids of very small height, of which the faces of {111} would be the bases.

Such faces are called *Vicinal faces*; and their arrangement accords with the general symmetry of the crystal as shown

'centre' of a crystal belonging to one of the above-mentioned ten classes is to some extent arbitrary.

* "Zeitschr. f. Kryst. u. Min.," xxxiv, 474.

† i.e. which seem to have the shape of a regular octahedron.

by Professor Miers in the case of alum *, and by C. Viola in the case of calcite †. This fact renders untenable the hypothesis of G. Wulff ‡ that they are due to a compromise between the molecular forces of the crystal trying to arrange particles deposited on it in faces having rational indices, and the forces due to the convection currents mentioned in § 4; for of course these convection currents do not partake of the symmetry of the crystal §.

§ 6. The arrangement of the vicinal faces may be a valuable help in determining the symmetry of a crystal in those cases where the presence of none but special forms makes it difficult to determine the symmetry otherwise. Since, however, they are not of universal occurrence, or at all easy to recognize, use is far more frequently made of the closely allied *etched-figures*.

When we expose a crystal face to the action of any solvent, it is found that solution does not take place uniformly over the whole surface, but that shallow pits are formed all over the faces, whose sides are formed of planes which are vicinal faces (see Fig. 60). These pits can be readily seen under the microscope ‖; they have the symmetry of the crystal; and it is only by means of these etched-figures that the symmetry of certain crystals has been determined.

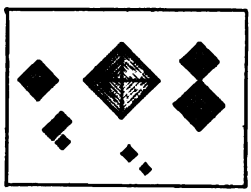

Fig. 60. Galena (class O_h) etched on a face perpendicular to a 4-al axis with cold HCl. (After F. Becke ¶).

§ 7. Crystals growing out of the same vapour or liquid do not grow independently, but exercise a mutual influence on one another.

If the deposition of solid matter takes place only slowly there is often a tendency on the part of the larger crystals to

* "Report of the British Association," 1894, p. 654.

† "Zeitschr. f. Kryst. u. Min.," xxxv, 232.

‡ "Zeitschr. f. Kryst. u. Min.," xxxiv, 461.

§ For though their direction and magnitude are determined partly by the molecular forces of the crystal and by the angles between its faces which partake of the symmetry of the crystal, they are also determined by the action of gravity which does *not* partake of this symmetry.

‖ The solvent should usually be applied for a very short time, and acids used as solvents should be fairly dilute; however, it is difficult to lay down general rules; experiments must be made to determine the best treatment for each kind of crystal.

¶ See Th. Liebisch's "Grundriss der physikalischen Krystallographie," p. 44.

grow at the expense of the smaller (which are redissolved); this is a result of the tendency towards a minimum of surface energy, as explained in § 2; for a given volume of solid has a smaller surface energy when it is collected into one big crystal, than when it is divided up into smaller ones *.

§ 8. Even, however, in those cases where the growth is too rapid, or some other reason prevents this 'eating up' of the smaller crystals by the larger, crystals may influence the growth of their neighbours. Thus we often have several crystals side by side, united along planes or partially interpenetrating, such that the corresponding faces of all the crystals are parallel. This arrangement is called *parallel grouping* (see Fig. 61); it is found sometimes even in the case of two crystals of *different* material but similar form.

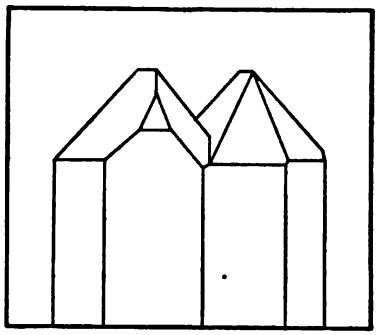

Fig. 61. Two quartz crystals in parallel position.

§ 9. Still more interesting is the case of two crystals which are not parallel, but which grow together in such a way that if one of them were to be rotated through an angle π about a certain line l, or reflected in a certain plane p, it would be parallel to the other. Such crystals are said to be 'twinned.'

The line l may be (1) normal to a possible face of both crystals; (2) parallel to a possible edge; (3) parallel to a possible face and perpendicular to a possible edge (the occurrence of this last case is, however, doubtful). The plane p must be parallel to a possible face.

Twinned crystals or *twins* may be united along a plane (called the 'composition plane'), and are then called 'contact,'

* Suppose we have two crystals of the same shape and of lengths a_1, a_2; and suppose that one 'eats up' the other, making one crystal of the same shape and of length b. Then $a_1^3 + a_2^3 = b^3$. Now the ratio of the surface energy of the single crystal to the sum of the surface energies of the two component crystals viz. $b^2 : a_1^2 + a_2^2$ is < 1; for

$$b^6 = (a_1^3 + a_2^3)^2 < a_1^6 + a_2^6 + 6a_1^3 a_2^3 . a_1 a_2 < a_1^6 + a_2^6 + 8a_1^3 a_2^3 (a_1^3 + a_2^3) < (a_1^2 + a_2^2)^3.$$

Hence so long as two crystals of the same sort exist together, the surface energy can be diminished by letting one eat up the other.

or 'juxtaposition twins' (Fig. 62); or they may partially or wholly interpenetrate, the surface separating them being of irregular shape, and are then called 'penetration twins' (Fig. 63). Very often we have a line or ring of crystals, the second twinned on the first and third, the third on the second

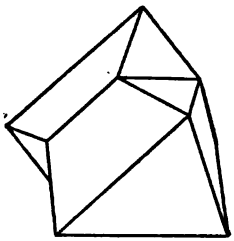

Fig. 62. Spinel twin* (class O_h; line l perpendicular to $(11\bar{1})$).

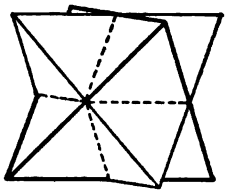

Fig. 63. Twin of two tetrahedra* (class T_d; line l perpendicular to (110)).

and fourth, the fourth on the third and fifth, and so on; this is often described as 'repeated twinning.'

In many cases re-entrant angles, &c., betray the existence of such a series of twins; sometimes, however, it is difficult to distinguish such a collection from a single crystal of a symmetry different from that of the individuals composing the series †.

§ 10. We have in chapters ii and x and in the present chapter given a short account of some of the more important properties which observation shows to be possessed by crystals. We shall devote the remainder of this book to a description of the theories suggested to account for these properties. Though these theories contain a certain speculative element, it must be borne in mind that they involve much that is capable of mathematical proof; and that, even if future experiment should show that some of the hypotheses laid down are untenable, yet the usefulness of the theories is not thereby destroyed, even though modification of their details may be necessary.

* Each crystal of the twin is bounded by faces of the form $\{111\}$; see p. 93, l. 12.

† We get thus so-called 'mimetic twinning.'

PART II

CHAPTER XII

THE STRUCTURE-THEORY.

§ 1. It is now generally assumed that *matter is not continuous but coarse-grained* *, i. e. that matter is composed of atoms which are practically indivisible and are situated at very small but not infinitesimal distances apart. In a solid these atoms may be grouped into 'molecules' (either chemical molecules or collections of these) which act as units in building up the solid.

Since the properties of a substance in the neighbourhood of a point within it are not in general affected by the position of the boundary of the substance, we can, and in future shall, consider the substance to be infinitely extended in every direction.

We must now modify our previous definition of homogeneity; we shall say that a substance is homogeneous if, when we take *any* two points A and B in it, a point C can always be found such that the physical behaviour of the substance about C is similar to its behaviour about A, and such that B and C (though not necessarily coincident) are so close together † that we cannot distinguish them even by the most refined methods of observation at our disposal.

It may happen that the geometrical positions of A and C with regard to the collection of molecules are always similar; in this case the substance is said to be crystalline. If, however, the physical properties of the substance about A and C, but *not* its geometrical properties, are always similar the substance is said to be amorphous.

These definitions imply the existence of a connexion between the orientation of various molecules building up a crystalline medium, which does not exist in the case of an amorphous (or 'vitreous') material.

On the fundamental assumptions of the coarse-grainedness of matter and on our defirition of a crystalline medium we

* See the Presidential address, British Association, 1901.

† A few millionths of a millimetre apart perhaps.

can build up a theory of the internal structure of crystals; or to speak more correctly, we can lay down certain rules to which all such theories must conform.

§ 2. If the collection of molecules composing a homogeneous crystalline medium (of infinite extent) has* a symmetry-operation it must possess* an infinite number of symmetry-operations; e. g. if it has an n-al rotation-axis it must have an infinite number of parallel rotation-axes—this follows from the definitions of homogeneity and crystalline matter in § 1. All the symmetry-operations of such a collection form a group† which contains an infinite number of operations; it is an 'infinite group.'

We shall now determine (chapters xiii, xiv, xvi–xxiii) the properties of all possible distinct groups of this sort; they are 230 in number as we shall prove later. Each group is completely defined when we have given the relative positions of all its elements of symmetry.

* A figure may be said to 'have' or 'possess' a symmetry-operation when it is brought to self-coincidence by that operation.

† This was proved on p. 47.

CHAPTER XIII

LATTICES AND TRANSLATIONS.

§ 1. A series of points on a straight line such that the distance between any two consecutive points of the series is constant and equal to a is called a 'regular row,' or more simply a *row* of points of interval a (Fig. 64).

A series of coplanar parallel lines such that the distance between any two consecutive lines of the series is constant and equal to a will be called a *set* of lines of interval a.

The sum total of the points of intersection of any two coplanar sets of lines is called a 'regular plane net,' or more simply a *net* (Fig. 65).

A series of parallel planes such that the distance between any two consecutive planes of the series is constant and equal to a will be called a *set* of planes of interval a.

The sum total of the points of intersection of any three sets of planes is called a 'regular space-lattice,' or more simply a *lattice* * (Fig. 66).

§ 2. The group of movements which brings the collection of molecules composing a solid homogeneous crystalline material to self-coincidence cannot contain an infinitesimal translation, for no two of the molecules are at an infinitesimal distance apart. Suppose that the *smallest* translation (if any) which the group possesses is a finite † translation T ‡ represented in magnitude and direction by OA_1 (Fig. 64). Take

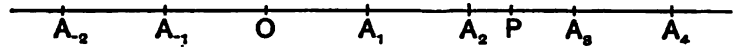

Fig. 64.

$A_1A_2 = A_2A_3 = \ldots = A_{-1}O = A_{-2}A_{-1} = \ldots = OA_1$; then T can equally well be represented by A_1A_2, A_2A_3, $-OA_{-1}$, $-A_1O$, or in fact by *any* straight line in the direction of and equal to OA_1.

* See note at the end of this chapter on p. 127.

† That is, 'neither infinite nor infinitesimal,' and so throughout this book.

‡ The group may have other translations equal to T but has none smaller than it.

The group must possess translations represented by OA_2, OA_3, ..., OA_{-1}, OA_{-2}, ... ; for since T is an operation of the group so are $T^2, T^3, \ldots, T^{-1}, T^{-2}, \ldots\ldots$ (p. 45). Moreover there is no translation of the group parallel to OA_1 not included in this series; for if there were such a translation T_1, represented by OP where P lay between A_n and A_{n+1} *, then $T_1 . T^{-n}$ would be a translation of the group represented by A_nP. This would be contrary to our hypothesis that T is the smallest translation of the group, since $A_nP < OA$.

If we represent OA_1 by $2\tau_1$ † ; $OA_2, OA_3, \ldots, OA_{-1}, OA_{-2}, \ldots$ are represented by $4\tau_1, 6\tau_1, \ldots, -2\tau_1, -4\tau_1, \ldots$.

It is often convenient to denote a translation by the same symbol as the line representing it; we talk of 'the translations $2\tau_1$, $4\tau_1$, $6\tau_1, \ldots$, $-2\tau_1$, $-4\tau_1, \ldots$' instead of 'the translations $T, T^2, T^3, \ldots, T^{-1}, T^{-2}, \ldots$'.

Such a translation as $2\tau_1$ or $-2\tau_1$, which is the smallest translation of the group in its own direction, is called a *primitive* translation in that direction.

§ 3. Now suppose that the group contains a translation T' whose direction is different from that of T, and which is not greater than any translation of the group not parallel to T‡. Let T' be represented in magnitude and direction by $OB_1 = 2\tau_2$. Along OB_1 take a 'row' of points $\ldots B_{-2}, B_{-1}, O, B_1, B_2, B_3, \ldots$ of interval OB_1, through them draw parallels to OA_1, and through $\ldots A_{-2}, A_{-1}, O, A_1, A_2, A_3, \ldots$ draw parallels to OB_1. These two 'sets' of lines meet at the points of a net (Fig. 65).

Let E be any point of this net, then OE represents a translation of the group; for if E be the intersection of the line through A_m parallel to OB_1, and the line through B_n parallel to OA_1, OE represents the resultant of the translations OA_m and A_mE, i.e. of OA_m and OB_n (since two parallel and equal lines represent the same translation), i.e. of $2m\tau_1$ and $2n\tau_2$.

There are no translations of the group parallel to the plane OA_1B_1 other than those represented by lines such as OE. For suppose there could be such a translation OV§, where V lies in (or on the perimeter of ‖) the parallelogram p formed

* If the point A_0 is supposed to coincide with O, this will cover all possible cases; n being zero or any positive or negative integer.

† The 2 is inserted to avoid fractions later.

‡ T' may be equal to T, but must not be smaller.

§ 'The translation OV' means 'the translation represented in magnitude and direction by OV,' and so throughout the book.

‖ But not at a vertex of p.

by the lines through A_{m-1} and A_m parallel to OB_1 and the lines through B_{n-1} and B_n parallel to OA_1. Then combining the translation OV with the translations T^{1-m} and T'^{1-n} we see that there must be a point F situated with respect to OA_1DB_1 in the same way that V is situated with respect

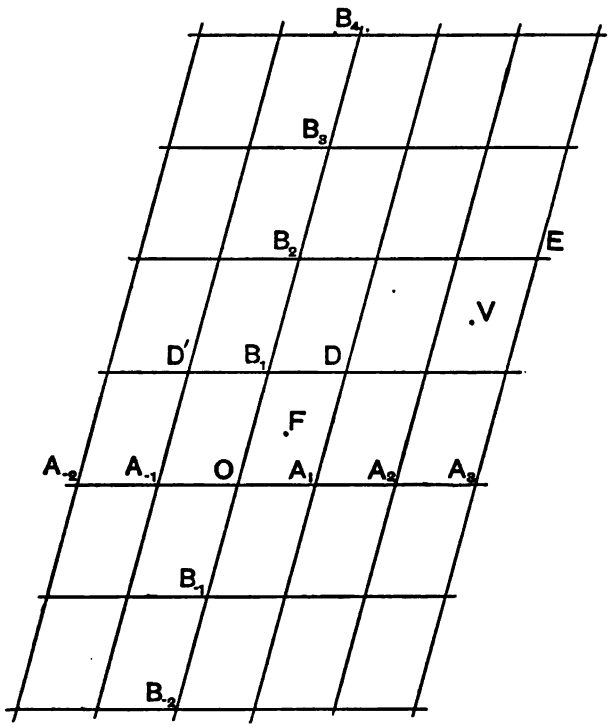

Fig. 65.

to p, and such that OF is a translation of the group. In this case the lines FA_1, FD, FB_1 would also represent translations of the group*, and would therefore be $\not< OB_1$ (and hence $\not< OA_1$). Consequently the angles subtended at F by all the sides of OA_1DB_1 would be $\not> \frac{\pi}{3}$. This however is impossible, for their sum is 2π.

If E is the intersection of a line through A_m parallel to OB_1, and a line through B_n parallel to OA_1, OE represents the resultant of the translation OA_m followed by the transla-

* For FA_1 would represent the resultant of the translations FO, OA_1, &c.

tion A_mE, i. e. of $2m\tau_1$ followed by $2n\tau_2$. We shall write this resultant in the form $2m\tau_1 + 2n\tau_2$. Similarly OE represents the resultant of the translation OB_n followed by the translation B_nE, i. e. of $2n\tau_2$ followed by $2m\tau_1$. We write this resultant in the form $2n\tau_2 + 2m\tau_1$. Then $2m\tau_1 + 2n\tau_2 = 2n\tau_2 + 2m\tau_1$, and hence the two translations $2m\tau_1$ and $2n\tau_2$ are permutable operations. An exactly similar proof leads us to the theorem:—

Any two translations are permutable operations.

We choose the addition sign instead of the multiplication to represent the resultant of two translations (if the translations are denoted by the same symbol as the lines representing them), in order to be consistent with the fact that the translation $2p\tau_1$ followed by the translation $2q\tau_1$ is evidently equivalent to the translation $2(p+q)\tau_1$.

We have proved that *any* translation of the group parallel to the plane OA_1B_1 can be represented by $2m\tau_1 + 2n\tau_2$ where m and n are positive or negative integers*; we therefore call $2\tau_1$ and $2\tau_2$ a *primitive pair* of translations 'in' or 'parallel to the plane' OA_1B_1. $2\tau_1$ and $2\tau_2$ are not the only primitive pair of translations parallel to this plane; for let $2\tau_1'$, $2\tau_2'$ be any two translations of the group parallel to the plane, then

$$2\tau_1' = 2p_1\tau_1 + 2p_2\tau_2$$
$$2\tau_2' = 2q_1\tau_1 + 2q_2\tau_2$$

where p_1, p_2, q_1, q_2 are integers.

Solving† we have

$$\tau_1 = \frac{q_2}{l}\tau_1' - \frac{p_2}{l}\tau_2'$$

$$\tau_2 = \frac{-q_1}{l}\tau_1' + \frac{p_1}{l}\tau_2'$$

where $l \equiv \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix}$ and is therefore integral.

If now $l = \pm 1$ (and by the theory of continued fractions it is possible to find an infinite number of integral solutions of the equation $p_1q_2 - p_2q_1 = \pm 1$) $2\tau_1$ and $2\tau_2$ are expressed in the form $2m\tau_1' + 2n\tau_2'$ (m and n integral); but by hypothesis any translation of the group parallel to the plane OA_1B_1 can

* Including zero.

† The justification of this step lies in the fact that $p\tau + q\tau = (p+q)\tau$ where τ is any translation, and in the fact that all translations are permutable.

be expressed in the form $2m\tau_1+2n\tau_2$, and therefore any translation parallel to this plane can be expressed in the form $2m\tau_1'+2n\tau_2'$; i.e. $2\tau_1', 2\tau_2'$ are a primitive pair.

Conversely, if $2\tau_1', 2\tau_2'$ are a primitive pair, $l=\pm1$. For in this case l must be a divisor of p_1, p_2, q_1, q_2 (otherwise $2\tau_1$ and $2\tau_2$ could not be expressed in the form $2m\tau_1'+2n\tau_2'$). Let therefore $p_1=p_1'l$, $p_2=p_2'l$, $q_1=q_1'l$, $q_2=q_2'l$, where $p_1', p_2', q_1', q_2', l$ are integers; then

$$l=\begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} = l^2\times\begin{vmatrix} p_1' & p_2' \\ q_1' & q_2' \end{vmatrix};$$

and therefore

$$1=l\times\begin{vmatrix} p_1' & p_2' \\ q_1' & q_2' \end{vmatrix} = l\times\text{an integer}.$$

$$\therefore\ l=\pm1.$$

The above reasoning holds if $2\tau_1, 2\tau_2$ represent *any* primitive pair (not necessarily OA_1, OB_1).

Since by the theory of continued fractions integers p_1, p_2 can be found to make $p_1q_2-p_2q_1=\pm1$ if q_1, q_2 are *any* given integers having no common factor; therefore in any net we can find a pair of primitive translations of which one is represented by the line joining O to *any given point* of the net E, provided no other point of the net lies on the finite straight line OE.

It is easily verified that $2\tau_1$, $2m\tau_1\pm2\tau_2$ (where m is any integer, and $2\tau_1, 2\tau_2$ are *any* primitive pair) form a primitive pair; in particular OA_1, B_1A_1 $(2\tau_1, 2\tau_1-2\tau_2)$ of Fig. 65 are such a pair. Hence *if one pair of sides of a triangle represents a primitive pair of translations, so also does any pair of sides of the triangle.*

§ 4. *To find the area of the triangle whose vertices are any three points of the net of* Fig. 65.

We may take one of the points as our origin of reference O; let a and b be the others.

Suppose $Oa=2p_1\tau_1+2p_2\tau_2$, $Ob=2q_1\tau_1+2q_2\tau_2$, where $OA_1=2\tau_1$, $OB_1=2\tau_2$ represent a primitive pair of translations in the plane OA_1B_1; and denote the angle A_1OB_1 by δ.

Let the coordinates of A_1, B_1 referred to any two rectangular axes through O in the plane OA_1B_1 be (x_1, y_1), (x_2, y_2); then the coordinates of a, b are

$$(p_1x_1+p_2x_2,\ p_1y_1+p_2y_2),\ (q_1x_1+q_2x_2,\ q_1y_1+q_2y_2),$$

respectively.

Then the area of Oab

$$= \tfrac{1}{2}\begin{vmatrix} 1 & p_1x_1+p_2x_2 & p_1y_1+p_2y_2 \\ 1 & q_1x_1+q_2x_2 & q_1y_1+q_2y_2 \\ 1 & 0 & 0 \end{vmatrix} = \tfrac{1}{2}\begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} \times \begin{vmatrix} 1 & x_1 & x_2 \\ 1 & y_1 & y_2 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} \times \text{the area of the triangle } OA_1B_1$$

$$= \tfrac{1}{2}\begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} OA_1 . OB_1 . \sin\delta.$$

Oa, Ob represent a primitive pair of translations if and only if

$$\begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} = \pm 1;$$

hence: *the areas of all triangles two of whose sides represent a primitive pair of translations in a given plane are equal: and conversely, if the area of a triangle D whose vertices are points of the net representing the translations in any plane is equal to that of a triangle, two of whose sides represent a primitive pair of translations in that plane, any two sides of D also represent a primitive pair.*

The above converse gives us at once another proof of the theorem that 'if one pair of sides of a triangle represents a primitive pair of translations, so also does any pair of sides of the triangle.'

§ 5. Let $OC_1 = 2\tau_3$ be the smallest of the translations of the group (if any) which is not parallel to the plane OA_1B_1. Along OC_1 construct a row of points $\ldots C_{-2}, C_{-1}, O, C_1, C_2, C_3, \ldots$ of interval OC_1, and through them draw planes parallel to OA_1B_1; through $\ldots A_{-2}, A_{-1}, O, A_1, A_2, A_3, \ldots$ draw planes parallel to OB_1C_1, and through $\ldots B_{-2}, B_{-1}, O, B_1, B_2, B_3, \ldots$ draw planes parallel to OC_1A_1. Then if E be *any* point of the lattice formed by the intersections of these three sets of planes, OE is a translation of the group* (Fig. 66), and is included among the translations $2m\tau_1 + 2n\tau_2 + 2s\tau_3$† (where m, n, s are integers). There are no other translations of the group; for if there were such a translation OV where V is not a point of the lattice, there must also be a translation of the group represented by OF, where F lies in or on the parallelepipedon, three of whose adjacent edges are OA_1, OB_1, OC_1*.

* The proof is exactly similar to that of the corresponding theorem of § 8.

† Now $2\tau_1 = OA_1$, $2\tau_2 = OB_1$.

In that case the lines joining F to all the vertices of this parallelepipedon would represent translations and would therefore be $\not< OC_1$; and consequently the angles subtended at F by all the sides of the parallelepipedon would be $\not> \frac{\pi}{3}$*.

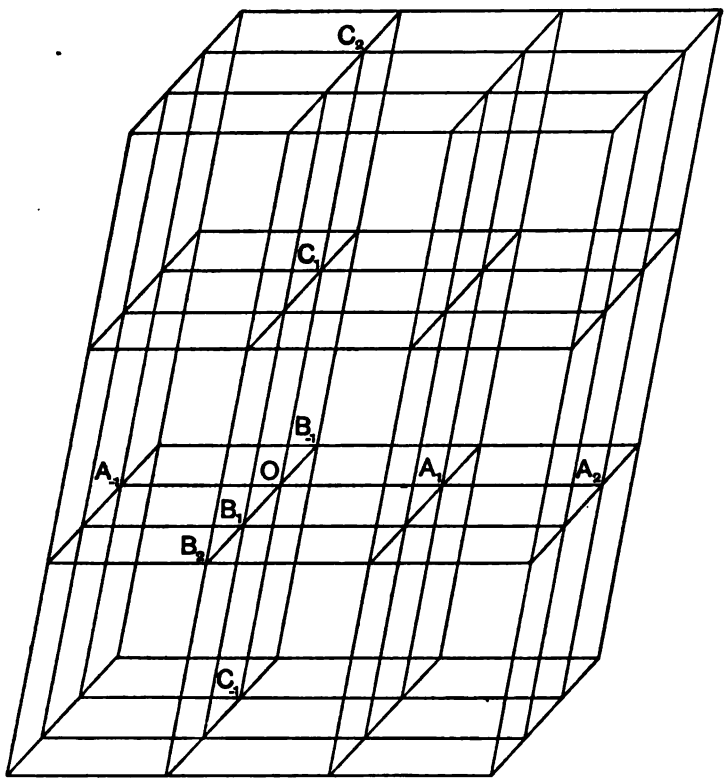

Fig. 66.

This however is impossible; for let the planes through F and the sides cut a sphere whose centre is F in the twelve edges of six foursided polygons covering the sphere. The area of at least one of these polygons must be $\not<$ a sixth of the area of the surface of the sphere, that is, $\not<$ the area of a regular foursided polygon whose sides are $\cos^{-1}\frac{1}{3}$. But the area of a foursided

* Cf. p. 117.

polygon whose sides are all $> \frac{\pi}{3}$ $(= \cos^{-1}\frac{1}{2})$ is evidently $<$ the area of *some* foursided polygon whose sides are $\cos^{-1}\frac{1}{3}$ *, and is therefore $<$ the area of the *regular* foursided polygon whose sides are $\cos^{-1}\frac{1}{3}$ †. Hence at least one of the angles subtended by the sides of the parallelepipedon at F must be $> \frac{\pi}{3}$.

We see then that every translation of a group (which contains only finite translations) can be expressed in the form $2m\tau_1 + 2n\tau_2 + 2s\tau_3$ where m, n, s are integers; $2\tau_1, 2\tau_2, 2\tau_3$ are therefore said to form a *primitive triplet* ‡.

There are an infinite number of primitive triplets; for let $2\tau_1', 2\tau_2', 2\tau_3'$ be any three translations of the group

then
$$\begin{aligned} 2\tau_1' &= 2p_1\tau_1 + 2p_2\tau_2 + 2p_3\tau_3 \\ 2\tau_2' &= 2q_1\tau_1 + 2q_2\tau_2 + 2q_3\tau_3 \\ 2\tau_3' &= 2r_1\tau_1 + 2r_2\tau_2 + 2r_3\tau_3 \end{aligned}$$

where p, q, r, &c., are integers.

Solving, for τ_1, τ_2, τ_3

we have
$$\begin{aligned} 2l\tau_1 &= 2P_1\tau_1' + 2Q_1\tau_2' + 2R_1\tau_3' \\ 2l\tau_2 &= 2P_2\tau_1' + 2Q_2\tau_2' + 2R_2\tau_3' \\ 2l\tau_3 &= 2P_3\tau_1' + 2Q_3\tau_2' + 2R_3\tau_3' \end{aligned}$$

where $l \equiv \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix}$, and P_1, Q_1, R_1, &c., are the cofactors of p_1, q_1, r_1, &c., in this determinant.

Now if $l = \pm 1$ (and we can determine $p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3$ in an infinite number of ways so as to satisfy this condition), $2\tau_1', 2\tau_2', 2\tau_3'$ are a primitive triplet; for since any translation can be expressed in the form $2m\tau_1 + 2n\tau_2 + 2s\tau_3$, it can, if $l = \pm 1$, be expressed in the form $2m\tau_1' + 2n\tau_2' + 2s\tau_3'$. Conversely if $2\tau_1', 2\tau_2', 2\tau_3'$ are a primitive triplet $l = \pm 1$. For since in this case l divides $P_1, Q_1, R_1, P_2, Q_2, R_2, P_3, Q_3, R_3$,

* Let $EFGH$ be a polygon whose sides are $> \frac{\pi}{3}$; then the area of the polygon $EF''GH''$ (where $EF'' = GF'' = EH'' = CH'' = \cos^{-1}\frac{1}{3}$) is $>$ that of the polygon $EF'GH'$ (where $EF' = GF' = \frac{1}{2}(EF + GF)$ and $EH' = GH' = \frac{1}{2}(EH + GH)$), and is therefore $>$ that of $EFGH$.

† This is a particular case of the theorem that 'the area of a spherical polygon, whose sides are given, is a maximum when the vertices all lie on a small circle.'

‡ German—ein primitives Tripel.

let $P_1 = P_1'l$, $P_2 = P_2'l$, $P_3 = P_3'l$, &c., where l, P_1', Q_1', R_1', &c., are integers.

Then $$l^3 = \begin{vmatrix} P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \\ P_3 & Q_3 & R_3 \end{vmatrix} = l^3 \times \begin{vmatrix} P_1' & Q_1' & R_1' \\ P_2' & Q_2' & R_2' \\ P_3' & Q_3' & R_3' \end{vmatrix},$$

and $$\therefore \quad 1 = l \times \begin{vmatrix} P_1' & Q_1' & R_1' \\ P_2' & Q_2' & R_2' \\ P_3' & Q_3' & R_3' \end{vmatrix} = l \times \text{an integer},$$

and $\therefore \quad l = \pm 1$.

The above reasoning holds if $2\tau_1$, $2\tau_2$, $2\tau_3$ are *any* primitive triplet (not necessarily OA_1, OB_1, OC_1).

We may easily verify that $2\tau_1$, $2\tau_2$, $2\tau_3 + 2m_1\tau_1 + 2m_2\tau_2$ are a primitive triplet (where m_1, m_2 are any integers, and $2\tau_1$, $2\tau_2$, $2\tau_3$ any primitive triplet); in particular we may easily verify that any three sides of the tetrahedron $OA_1B_1C_1$ (Fig. 66) form a primitive triplet; hence:—*If any three non-coplanar sides of a tetrahedron represent a primitive triplet of a group of translations; the same is true of any other three non-coplanar sides of the tetrahedron.*

§ 6. To find the conditions that any two translations both parallel to a plane h may be a primitive pair parallel to h.

We shall suppose any translation represented by a line through the origin O*. Let then the two translations be $2q_1\tau_1 + 2q_2\tau_2 + 2q_3\tau_3 \equiv 2\tau_2'$, $2r_1\tau_1 + 2r_2\tau_2 + 2r_3\tau_3 \equiv 2\tau_3'$, where $2\tau_1$, $2\tau_2$, $2\tau_3$, are any primitive triplet. Then if any other translation parallel to h is $2m_1\tau_1 + 2m_2\tau_2 + 2m_3\tau_3 \equiv 2\tau_1'$, we must have

$$\left.\begin{aligned} m_1 &= \lambda q_1 + \mu r_1 \\ m_2 &= \lambda q_2 + \mu r_2 \\ m_3 &= \lambda q_3 + \mu r_3 \end{aligned}\right\}, \text{ since } \begin{vmatrix} m_1 & q_1 & r_1 \\ m_2 & q_2 & r_2 \\ m_3 & q_3 & r_3 \end{vmatrix} = 0$$

if $2\tau_1'$, $2\tau_2'$, $2\tau_3'$ are coplanar.

Solving, we have

$$\lambda = \frac{m_2r_3 - m_3r_2}{q_2r_3 - q_3r_2} = \frac{m_3r_1 - m_1r_3}{q_3r_1 - q_1r_3} = \frac{m_1r_2 - m_2r_1}{q_1r_2 - q_2r_1}.$$

If $q_2r_3 - q_3r_2$, $q_3r_1 - q_1r_3$, $q_1r_2 - q_2r_1$ have no common factor but unity, these equations can only hold good when λ is integral. Similarly μ must be integral. Therefore any translation $2\tau_1'$ in the plane of $2\tau_2'$, $2\tau_3'$ can be expressed in the form

* This is lawful, for all lines of the same length and direction represent the same translation.

$2\lambda\tau_2' + 2\mu\tau_3'$ where λ and μ are integers; i.e. $2\tau_2'$ and $2\tau_3'$ form a primitive pair parallel to the plane h.

Now suppose $q_2r_3 - q_3r_2$, $q_3r_1 - q_1r_3$, $q_1r_2 - q_2r_1$ have a prime common factor d. If q_1, q_2, q_3 have a common factor $2\tau_2'$ is not a primitive translation in its own direction and therefore evidently $2\tau_2'$, $2\tau_3'$ cannot form a primitive pair parallel to h*. Suppose q_1, q_2, q_3 have no common factor, then one of them, say q_1, is prime to d.

Find an integer L such that $Lq_1 + Mr_1$ is divisible by d, M being any integer prime to d †.

. Then $q_1(Lq_2 + Mr_2) = q_2(Lq_1 + Mr_1) + (q_1r_2 - q_2r_1)M$
and $q_1(Lq_3 + Mr_3) = q_3(Lq_1 + Mr_1) + (q_1r_3 - q_3r_1)M$
are both divisible by d; and therefore $Lq_2 + Mr_2$, $Lq_3 + Mr_3$ are both divisible by d, for q_1 is prime to d.

Put $L = \lambda d$, $M = \mu d$. Then we see that quantities λ, μ can be chosen, such that one at least is not an integer ‡, and such that $\lambda q_1 + \mu r_1$, $\lambda q_2 + \mu r_2$, $\lambda q_3 + \mu r_3$ are integers. In this case the translation $2\lambda\tau_2' + 2\mu\tau_3'$ belongs to the group, and it is evidently parallel to the plane h.

But λ and μ are *not* both integral, and therefore $2\tau_2'$, $2\tau_3'$ are not a primitive pair. Hence:—

The translations $2q_1\tau_1 + 2q_2\tau_2 + 2q_3\tau_3$, $2r_1\tau_1 + 2r_2\tau_2 + 2r_3\tau_3$ *do or do not form a primitive pair in the plane in which they lie according as* $q_2r_3 - q_3r_2$, $q_3r_1 - q_1r_3$, $q_1r_2 - q_2r_1$ *have not or have a common factor* ‖.

If $2\tau_2'$, $2\tau_3'$ are a primitive pair in their own plane then there is a translation 2τ of the group, such that 2τ, $2\tau_2'$, $2\tau_3'$ form a primitive triplet.

For by the theory of continued fractions it is always possible to find integers p_1, p_2, p_3 such that

$$\begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} = p_1(q_2r_3 - q_3r_2) + p_2(q_3r_1 - q_1r_3) + p_3(q_1r_2 - q_2r_1) = 1$$

if $q_2r_3 - q_3r_2$, $q_3r_1 - q_1r_3$, $q_1r_2 - q_2r_1$ have no common factor.

§ 7. *To find the volume of a tetrahedron whose vertices are any four points of the lattice of* Fig. 66.

* See p. 119.

† This is always possible, for q_1, $2q_1$, $3q_1, \ldots (d-1)q_1$ leave different remainders when divided by d, since q_1 is prime to d.

‡ μ is not an integer, for M was taken prime to d.

‖ This proof is due to Prof. E. B. Elliott. It may be noted that if $q_1r_2 - q_2r_1$ and $q_2r_1 - q_1r_2$ are both divisible by d which is prime to q_1, then of necessity $q_2r_3 - q_3r_2$ is divisible by d, for $q_1(q_2r_3 - q_3r_2) + q_2(q_3r_1 - q_1r_3) + q_3(q_1r_2 - q_2r_1) \equiv 0$.

We may take one of the points as our origin of reference O; let a, b, and c be the other three.

Suppose
$$\begin{aligned} Oa &= 2p_1\tau_1 + 2p_2\tau_2 + 2p_3\tau_3 \\ Ob &= 2q_1\tau_1 + 2q_2\tau_2 + 2q_3\tau_3 \\ Oc &= 2r_1\tau_1 + 2r_2\tau_2 + 2r_3\tau_3 \end{aligned}$$

where $OA_1 = 2\tau_1$, $OB_1 = 2\tau_2$, $OC_1 = 2\tau_3$ represent any primitive triplet of translations; and let OA_1, OB_1, OC_1 make angles α, β, γ with one another.

Let the coordinates of A_1, B_1, C_1 referred to any three orthogonal axes through O be (x_1, y_1, z_1) or $(OA_1.l_1, OA_1.m_1, OA_1.n_1)$; (x_2, y_2, z_2) or $(OB_1.l_2, OB_1.m_2, OB_1.n_2)$; and (x_3, y_3, z_3) or $(OC_1.l_3, OC_1.m_3, OC_1.n_3)$ respectively. Then the coordinates of a, b, c are

$$\begin{aligned} &(p_1x_1 + p_2x_2 + p_3x_3,\ p_1y_1 + p_2y_2 + p_3y_3,\ p_1z_1 + p_2z_2 + p_3z_3), \\ &(q_1x_1 + q_2x_2 + q_3x_3,\ q_1y_1 + q_2y_2 + q_3y_3,\ q_1z_1 + q_2z_2 + q_3z_3), \\ &(r_1x_1 + r_2x_2 + r_3x_3,\ r_1y_1 + r_2y_2 + r_3y_3,\ r_1z_1 + r_2z_2 + r_3z_3) \end{aligned}$$

respectively.

Therefore the volume of the tetrahedron $O\,abc$

$$= \frac{1}{6} \times \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & p_1x_1 + p_2x_2 + p_3x_3 & p_1y_1 + p_2y_2 + p_3y_3 & p_1z_1 + p_2z_2 + p_3z_3 \\ 0 & q_1x_1 + q_2x_2 + q_3x_3 & q_1y_1 + q_2y_2 + q_3y_3 & q_1z_1 + q_2z_2 + q_3z_3 \\ 0 & r_1x_1 + r_2x_2 + r_3x_3 & r_1y_1 + r_2y_2 + r_3y_3 & r_1z_1 + r_2z_2 + r_3z_3 \end{vmatrix}$$

$$= \frac{1}{6} \times \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} \times \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

$$= \frac{OA_1.OB_1.OC_1}{6} \times \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} \times \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}$$

$$= \frac{OA_1.OB_1.OC_1}{6} \times \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} \times \left\{ \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \right\}^{\frac{1}{2}}$$

$$= \frac{OA_1.OB_1.OC_1}{6} \times \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} \times \left\{ \begin{vmatrix} l_1^2 + m_1^2 + n_1^2 & l_1l_2 + m_1m_2 + n_1n_2 & l_3l_1 + m_3m_1 + n_3n_1 \\ l_1l_2 + m_1m_2 + n_1n_2 & l_2^2 + m_2^2 + n_2^2 & l_2l_3 + m_2m_3 + n_2n_3 \\ l_3l_1 + m_3m_1 + n_3n_1 & l_2l_3 + m_2m_3 + n_2n_3 & l_3^2 + m_3^2 + n_3^2 \end{vmatrix} \right\}^{\frac{1}{2}}$$

$$= \frac{OA_1 . OB_1 . OC_1}{6} \times \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} \times \left\{ \begin{vmatrix} 1 & \cos\gamma & \cos\beta \\ \cos\gamma & 1 & \cos\alpha \\ \cos\beta & \cos\alpha & 1 \end{vmatrix} \right\}^{\frac{1}{2}}.$$

Now we note that if through any point P lines PA, PB, PC are drawn equal and parallel to any three non-coplanar edges (e. g. A_1C_1, OB_1, OC_1) of a tetrahedron $OA_1B_1C_1$, then the tetrahedra $PABC$, $OA_1B_1C_1$ are equal; for they stand on equal bases (PBC, OB_1C_1) and are evidently of equal altitude.

Therefore, since Oa, Ob, Oc represent a primitive triplet, if and only if $\begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} = \pm 1$, we have the following theorem:—

The volumes of all tetrahedra, three of whose non-coplanar sides represent a primitive triplet of translations, are equal: and conversely, if the volume of a tetrahedron D, whose vertices are points of the lattice representing the group of translations, is equal to that of a tetrahedron three of whose non-coplanar sides represent a primitive triplet; any three non-coplanar sides of D also represent a primitive triplet.

The above converse gives us at once another proof of the theorem that 'if any three non-coplanar sides of a tetrahedron represent a primitive triplet, the same is true of any other three non-coplanar sides of the tetrahedron.'

§ 8. If O, a, b, c be any points of a lattice such that Oa, Ob, Oc represent a primitive triplet, then Ob, Oc must represent a primitive pair of translations parallel to the plane Obc. For if they do not, let Ob', Oc' represent a primitive pair in that plane. Then the triangle $Ob'c' < Obc$, and therefore the tetrahedron $AOb'c' < AObc$ which is impossible.

A similar result holds for Oc, Oa, and for Oa, Ob.

Again Oa must be a primitive translation in its own direction. For if it is not let Oa' be the primitive translation in that direction. Then $Oa' < Oa$ and therefore the tetrahedron $a'Obc < aObc$ which is impossible.

A similar result holds for Ob and Oc.

§ 9. In § 5 we arrived at a primitive triplet by taking A_1 as near O as possible; B_1 as near O as is consistent with the distinctness of the lines OA_1, OB_1; and C_1 as near O as is possible when points in the plane OA_1B_1 are excluded. We proved then that OA_1, OB_1, OC_1 form a primitive triplet.

We can also arrive at a primitive triplet as follows:—Take

any point A of the lattice such that OA represents a primitive translation in its own direction. In *any* net whose plane passes through OA take a point B which is no farther from OA than any other point of this net not lying in OA. Then OA, OB are a primitive pair in their own plane; for it is possible to find *some* point D in the net such that OA, OD are a primitive pair (p. 119), and D cannot be farther from OA than B, for then we should have the triangle $ODA > OBA$.

Now take a point C of the lattice as near the plane OAB as possible, but not lying in this plane. Then OA, OB, OC represent a primitive triplet, for it is possible to find some point E such that OA, OB, OE have this property (p. 124), and E cannot be farther from OAB than C, for then we should have the tetrahedron $EOAB > COAB$.

There is a large amount of choice in this way of finding a primitive triplet; for even when we have fixed OA and the plane of OAB we have an infinite number of points which may be taken as B (lying along two lines parallel to OA), and an infinite number of points which may be taken as C (lying in two planes parallel to OAB).

NOTE ON TERMINOLOGY.

The following is a comparison of words used by various authors in this subject:—

English.	*French.*	*German.*
Row	Rangée	Punktreihe.
Net	Réseau	Punktnetz.
Lattice	Assemblage	Raumgitter.
Point (of net or lattice)	Sommet	Punkt.

The words 'lattice, assemblage, Raumgitter' are not satisfactory; 'assemblage' is too vague, while 'lattice' suggests a *plane* figure. We have, however, employed it as shorter than 'space-lattice,' which is in more general use, but which does not suggest a three-dimensional figure in spite of the prefix 'space.'

CHAPTER XIV

SYMMETRICAL LATTICES.

§ 1. We have now proved that every translation of any group can be represented by straight lines drawn from any point of a certain lattice to every other point of the lattice. If we have any two translations of the group

$$T = 2p_1\tau_1 + 2p_2\tau_2 + 2p_3\tau_3,\ T' = 2q_1\tau_1 + 2q_2\tau_2 + 2q_3\tau_3,$$

then $T.T' = 2(p_1+q_1)\tau_1 + 2(p_2+q_2)\tau_2 + 2(p_3+q_3)\tau_3$ is also a translation, and belongs to the group; hence *the translations of any group form a subgroup.*

We shall call this subgroup the *translation-group.* Owing to the fact that the subgroup formed by the translations of any group can be represented by a lattice, it is of importance to determine the various kinds of symmetry possible in a lattice; we shall begin by considering the symmetry of 'nets.'

§ 2. Consider a net which has one of its points O fixed (Fig. 65); in this case any symmetry-element of the net must be a rotation-axis, axis of rotatory-reflexion, or symmetry-plane passing through O.

Now evidently *every* net with one point O fixed has as symmetry-elements (1) a centre of symmetry at O, (2) a 2-al rotation-axis through O perpendicular to the plane of the net, (3) a symmetry-plane coinciding with the plane of the net *.

In general the net has † no other symmetry-operations leaving O unmoved, but in particular cases it may possess a 2-al rotation-axis lying in the plane of the net, or a 4-al or 6-al rotation-axis perpendicular to this plane. Evidently a net can possess no symmetry-axis of the first sort other than these ‡; if a net has an n-al symmetry-axis of the second

* Any two of these symmetry-elements involves the third.

† See footnote, p. 114.

‡ Rotation about any other rotation-axis not perpendicular to the plane of the net does not bring the plane of the net, and therefore does not bring the net to self-coincidence. If there is an n-al rotation-axis perpendicular to the plane n may be taken even (for *every* net has a 2-al rotation-axis perpendicular to its plane). Let a rotation through $\frac{2\pi}{n}$ about this axis bring A_1 to A. Then by hypothesis $A_1A < OA_1$; $\therefore\ n > 6$.

sort it must also possess an n-al or $2n$-al rotation-axis owing to the existence of the centre of symmetry, so that a separate discussion is unnecessary. (In particular if the net has a symmetry-plane it has a perpendicular 2-al rotation-axis).

§ 3. Suppose that the net has a 2-al rotation-axis l (passing through O) lying in its own plane; then since the net has a 2-al rotation-axis perpendicular to its plane, by Euler's construction it must have another 2-al rotation-axis l' lying in its plane perpendicular to l.

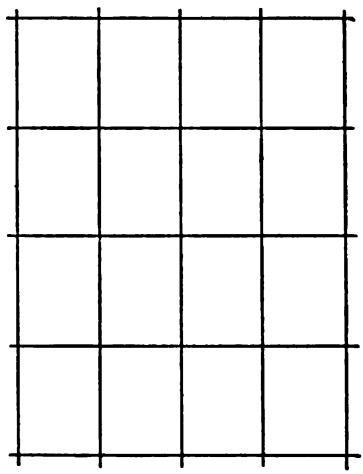

Fig. 67. Rectangular net.

First suppose OA_1 perpendicular to OB_1 in Fig. 65. Then evidently the net has two 2-al rotation-axes lying in its plane, namely, the lines OA_1, OB_1.

In this case the net is said to be *rectangular* (Fig. 67).

Next suppose OA_1 not perpendicular to OB_1.

First let l (or l') be perpendicular to OA_1, and let a rotation through π about l bring OB_1 into the position OB, then B must be a point of the net. Also B_1B is parallel to OA_1,

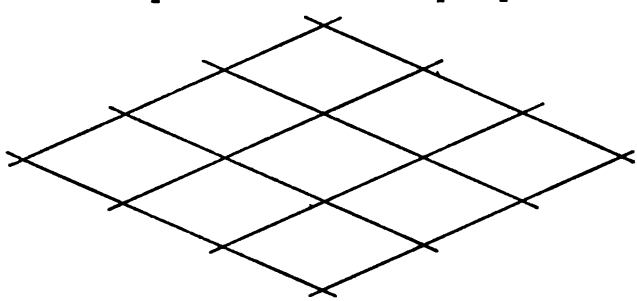

Fig. 68. Rhombic net.

and therefore B must coincide with one of the points of the net which lie on B_1D (Fig. 65). But we know that $OB_1 = O$B is the shortest distance between two points of the net (unless the two points lie on a line parallel to OA_1); therefore B must

coincide with D'*. Now OB_1, OD' form a primitive pair of translations of the group represented by the net, since the triangles OA_1B_1, OB_1D' are equal (p. 120); hence the group has a primitive pair of *equal* translations. In this case the net is said to be *rhombic* (Fig. 68).

Again, let neither l nor l' be perpendicular to OA_1; and let A_1 be brought to A by rotation about l. Then in our construction of the net of Fig. 65 we can take OB_1 as coinciding with OA, so that we arrive again at a rhombic set.

§ 4. Suppose that the net has a 4-al rotation-axis l perpendicular to its plane, and let A_1 be brought to A by a rotation through $\frac{\pi}{2}$ about l.

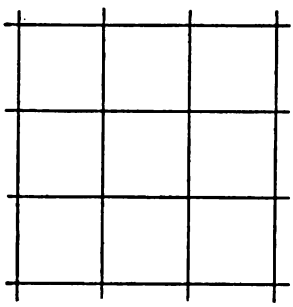

Fig. 69. Square net.

Then we may take OB_1 as coinciding with OA, and arrive at a net with square meshes which we shall call a *square* net. (Fig. 69).

Suppose that the net has a 6-al rotation-axis l perpendicular to its plane, and let A_1 be brought to A, A by rotations through $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ about l. Then we may take OB_1 as coinciding with OA or OA, and arrive at a net formed by rhombi with angles $\frac{\pi}{3}$ and $\frac{2\pi}{3}$. Such a net is called *equilateral* (Fig. 70).

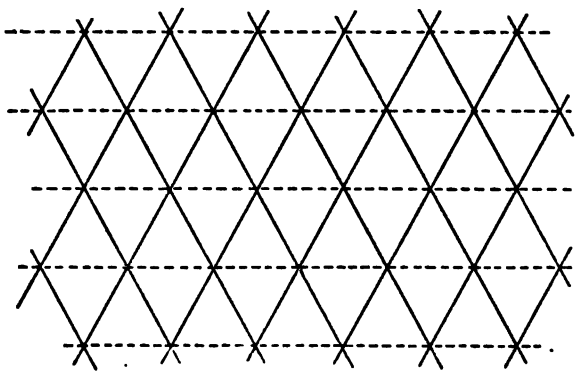

Fig. 70. Equilateral net.

* We suppose the angle A_1OB_1 acute, as is evidently lawful.

The operations which bring to self-coincidence a rectangular, rhombic, square, or equilateral net, but which leave one of the points unmoved, form the groups D_{2h}, D_{2h}, D_{4h}, or D_{6h} respectively.

§ 5. We now proceed to consider those symmetry-operations of any lattice which leave one point unmoved. As before, it is only necessary to consider operations of the first sort; for every point of a lattice is evidently a centre of symmetry of that lattice. If a lattice has a system of rotation-axes through one of its points, it has a similar and similarly orientated system through each of its points; for any translation of the group represented by the lattice is evidently a symmetry-operation of that lattice, and any symmetry-operation of a figure brings the system formed by its symmetry-elements to self-coincidence (p. 47).

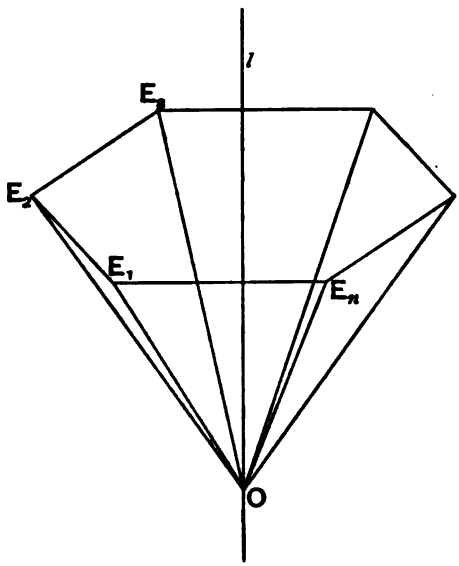

Fig. 71.

If the symmetry-operations of a lattice which leave one point O unmoved form a group G, the lattice will be said to 'have the symmetry of the group G.'

The perfectly general lattice has no rotation-axes; it has the symmetry of the group C_i, and is denoted by Γ_{ii}.

Now suppose that the lattice has an n-al rotation-axis l through O, and let E_1 be any point of the lattice. Then if rotations through $\frac{2\pi}{n}, \frac{4\pi}{n}, \frac{6\pi}{n}, \ldots, \frac{2(n-1)\pi}{n}$ about l bring E_1 into the positions $E_2, E_3, E_4, \ldots, E_n$, these points are also points of the lattice which lie in a net perpendicular to l (Fig. 71). Therefore:—

Every rotation-axis of a lattice has a net perpendicular to it.*

* The proof fails if $n = 2$; in this case let any points E_1, F_1, G_1,... of the

Again :—

Every symmetry-plane of a lattice which contains one point of the lattice, contains a net of points;

for the symmetry-plane has a 2-al rotation-axis perpendicular to it, since O is a centre of symmetry of the lattice.

Since the resultant of all the translations represented by $OE_1, OE_2, OE_3, \ldots, OE_n$ is parallel to l, therefore :—

Every rotation-axis of a lattice passing through O passes through an infinite number of points of the lattice.*

If an n-al rotation-axis l passes through O (Fig. 71), a parallel n-al rotation-axis passes through E_1. Rotation through $\frac{2\pi}{n}$ about this latter axis must evidently bring the net in the plane $E_1 E_2 \ldots E_n$ to self-coincidence since it brings the whole lattice to self-coincidence.

But a rotation-axis of a net passing through a point of the net can only be a 2-al, 4-al, or 6-al axis. Therefore :—

A rotation-axis of a lattice which passes through one of its points is 2-al, 3-al, 4-al, or 6-al.

§ 6. We shall always denote a primitive triplet of the translations represented by a lattice by $2\tau_1, 2\tau_2, 2\tau_3$.

In the lattice Γ_{t_r} (whose symmetry is that of C_i) the translations forming a primitive triplet are independent of one another in magnitude and direction.

If a lattice has a 2-al rotation-axis this axis must be parallel to a row of the lattice. Let C be a point on this row next to O, then OC $(= 2\tau_z)$ may be taken as one of a triplet of primitive translations of the lattice. Let OX $(= 2\tau_1)$ and OY $(= 2\tau_2)$ be the two other members of the triplet.

Let ϵ_1, ϵ_2 be the two nets in the planes COX, COY; then a primitive pair of translations of ϵ_1 is OX, OC and of ϵ_2 is OY, OC.

Now ϵ_1, ϵ_2 must be brought to self-coincidence by a rotation through π about the axis OC lying in their plane (for the lattice as a whole is brought to self-coincidence by such a rotation); therefore these nets are either rectangular or rhombic.

lattice be brought to $E_2, F_2, G_2, \ldots$ by a rotation through π about l. Then $E_1E_2, F_1F_2, G_1G_2, \ldots$ represent translations of the group perpendicular to l. Hence if E_1F' is parallel and equal to F_1F_2, E_1G' to G_1G_2, &c., the points $E_1, E_2, F', G' \ldots$ belong to the lattice and lie in a net perpendicular to l.

* Of course a straight line passing through two points of a lattice passes through an infinite number of points of the lattice.

In all cases we see from Figs. 67, 68 that there are translations in the nets ϵ_1, ϵ_2 perpendicular to OC. We shall denote these translations by $2\tau_e$, $2\tau_f$. If ϵ_1 is rectangular $OX = 2\tau_e$, if ϵ_1 is rhombic $OX = \tau_e + \tau_g$; and similarly in the case of ϵ_2.

If ϵ_1, ϵ_2 are both rectangular OC is perpendicular to both OX and OY, but OX is not in general perpendicular to OY. The lattice in this case may be considered as formed by placing points at the vertices of a series of right prisms with parallelogrammatic bases, which are similar and similarly situated and fill all space *. We denote this kind of lattice by Γ_m; for it

$$2\tau_1 = 2\tau_e; \quad 2\tau_2 = 2\tau_f, \quad 2\tau_3 = 2\tau_g \quad . \quad . \quad . \quad (1).$$

If ϵ_1 is rectangular and ϵ_2 rhombic,

$$2\tau_1 = 2\tau_e, \quad 2\tau_2 = \tau_f + \tau_g, \quad 2\tau_3 = 2\tau_g \quad . \quad . \quad (2),$$

and the case of ϵ_1 rhombic and ϵ_2 rectangular is of course geometrically indistinguishable from this.

If ϵ_1 and ϵ_2 are both rhombic,

$$2\tau_1 = \tau_e + \tau_g, \; 2\tau_2 = \tau_f + \tau_g, \; 2\tau_3 = 2\tau_g. \quad . \quad . \quad (3).$$

We shall now show that cases (2) and (3) are geometrically indistinguishable.

In Fig. 72 let $OA = 2\tau_e$, $OB = 2\tau_f$; bisect BC, CA, AB at A_1, B_1, C_1†; and let C_2 be a point such that $OC_2 = \tau_e - \tau_f$.

Taking case (3) we have OX, OY coinciding with OB_1, OA_1; so that OB_1, OA_1, OC represent a primitive triplet.

Now the tetrahedron OCA_1C_2 = the tetrahedron OCA_1B_1, for they are on the same base and between the same parallels.

Therefore, since three non-coplanar edges of OCA_1B_1 (i.e. OB_1, OA_1, OC) represent a primitive triplet, OC_2, OA_1, OC, which are three non-coplanar edges of OCA_1C_2, also represent a primitive triplet.

Therefore OC_2, OC is a primitive pair in the plane OCC_2; and the net OCC_2 is rectangular. We are thus led back to case (2).

We denote this kind of lattice (2) or (3) by Γ_m'; and proceed to find various primitive triplets belonging to it. We no longer confine ourselves to the case where OC is one of the triplet; but we still take $2\tau_e$ and $2\tau_f$ as the translations

* That is to say, situated in the same way as the parallelepipeda of Fig. 66; each vertex (or angular point) is common to eight of the parallelepipeda.

† These symbols will not in future necessarily denote the same points as in Fig. 66.

perpendicular to OC and lying in planes through OC and two of the primitive triplet; and still denote OC by $2\tau_z$, and the primitive triplet by $2\tau_1, 2\tau_2, 2\tau_3$.

Since the triangle $A_1OB = A_1OC$; therefore the tetrahedra C_2A_1OB, C_2A_1OC are equal, and hence OA_1, A_1B, OC_2 represent a primitive triplet

$$2\tau_1 = \tau_f + \tau_z,\ 2\tau_2 = \tau_f - \tau_z,\ 2\tau_3 = 2\tau_\theta \quad . \quad . \quad . \quad (4).$$

The lattice may be considered as formed by placing points at the vertices of a series * of prisms with rhombic bases and axes perpendicular to one of the diagonals of the base but not to the other ($OA_1BC_2B_1C_1$ is half such a prism).

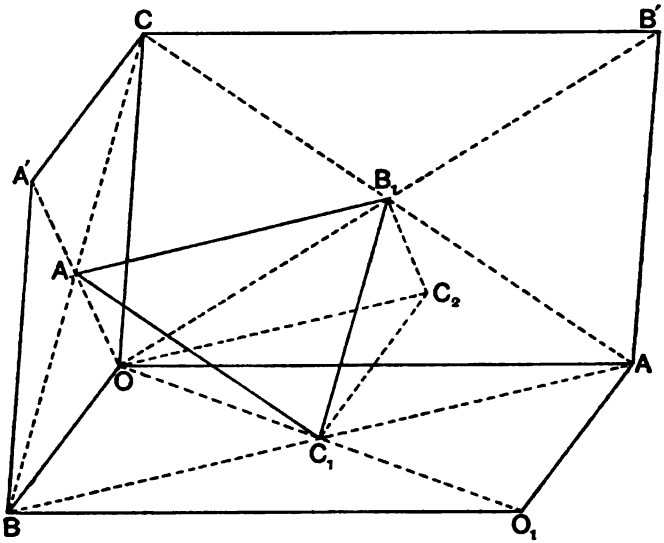

Fig. 72.

Again, Fig. 72 shows that the lattice may be formed by placing points at the vertices of a series of right prisms (of height OC) with parallelograms (such as OC_2C_1B) for their bases, and at the centres (A_1, B_1) of two opposite side-faces of the prisms.

Since the tetrahedra $OA_1B_1C_1$, OA_1B_1C are equal, therefore OA_1, OB_1, OC_1 represent a primitive triplet

$$2\tau_1 = \tau_f + \tau_z,\ 2\tau_2 = \tau_z + \tau_\theta,\ 2\tau_3 = \tau_\theta + \tau_f \quad . \quad . \quad (5).$$

Hence the lattice may be formed by placing points at the

* See footnote *, p. 133.

vertices and at the centres (A_1, B_1, C_1, &c.) of *all* the faces of right prisms (of height OC) with parallelograms (such as OAO_1B) as their bases.

Again $OB_1C_1C_2 = AB_1C_2C_1 = C_1A_1OB = B_1A_1OC$.

Therefore OC_1, OC_2, OB_1 represent a primitive triplet.

$$2\tau_1 = 2\tau_e,\ 2\tau_2 = 2\tau_f,\ 2\tau_3 = \tau_e + \tau_f + \tau_z \quad . \quad . \quad (6).$$

Hence the lattice may be formed by placing points at the vertices and at the centres (B_1) of right prisms (of height OC) with parallelograms (such as OC_1AC_2) as their bases.

Both the lattices described in this section have the symmetry of the group C_{2h}, for they both have a centre of symmetry at O, and a 2-al rotation-axis passing through O.

§ 7. Suppose now that the lattice has three mutually perpendicular rotation-axes OX, OY, OZ passing through O. Each of these axes passes through other points of the lattice; let $2\tau_x$, $2\tau_y$, $2\tau_z$ be the primitive translations of the group represented by the lattice in the directions of these axes.

One type of lattice (Γ_o) has these translations as a primitive triplet

$$2\tau_1 = 2\tau_x,\ 2\tau_2 = 2\tau_y,\ 2\tau_3 = 2\tau_z \quad . \quad . \quad . \quad . \quad (1).$$

This lattice may be formed by putting points at the vertices of a series of right prisms on rectangular bases; i.e. at the vertices of a series of right-angled parallelepipeda.

Now let any translation of any group represented by a lattice with OX, OY, OZ as three 2-al rotation-axes be $\tau + \tau'$, where τ, τ' are the components of the translation parallel and perpendicular to OZ. Then the group contains the translation $\tau - \tau'$, since OZ is a 2-al axis of the lattice. Therefore $2\tau = (\tau + \tau') + (\tau - \tau')$ is a translation of the group, so that τ is a multiple of τ_z.

Similarly, the component of any translation parallel to OX is a multiple of τ_x, and the component parallel to OY is a multiple of τ_y. Therefore every translation of the lattice is of the form $m\tau_x + n\tau_y + p\tau_z$.

Combining any translation $m\tau_x + n\tau_y + p\tau_z$ with multiples of $2\tau_x$, $2\tau_y$, $2\tau_z$, we see that every lattice with three orthogonal 2-al axes other than Γ_o must have one or more of the translations $\tau_y + \tau_z$, $\tau_z + \tau_x$, $\tau_x + \tau_y$, $\tau_x + \tau_y + \tau_z$; and all possible lattices are found by combining one or more of these with $2\tau_x$, $2\tau_y$, $2\tau_z$.

Any net through a rotation-axis, and in particular each of the nets in the planes YOZ, ZOX, XOY must be either rhombic or rectangular. If one of them (say that in the plane

XOY) is rhombic, $\tau_x+\tau_y$, $\tau_x-\tau_y$ are a primitive pair in the plane of this net. The lattice cannot in this case have the translation $\tau_x+\tau_y+\tau_z$, for then it would have

$$(\tau_x+\tau_y+\tau_z)-(\tau_x+\tau_y)=\tau_z$$

contrary to hypothesis. If it contains neither $\tau_x+\tau_z$ nor $\tau_y+\tau_z$ the nets in the planes ZOX, ZOY are rectangular, and a primitive triplet of the lattice is

$$2\tau_1=\tau_x+\tau_y,\ 2\tau_2=\tau_x-\tau_y,\ 2\tau_3=2\tau_z \quad . \quad . \quad (2).$$

This lattice (Γ_o') may be formed by placing points at the vertices of a series of right prisms with rhombic bases (base OC_2AC_1, height OC in Fig. 72*); or by placing points at the vertices and at the centres of the bases of right prisms with rectangular bases (base OAO_1B, height OC).

If, however, a lattice for which the net in the plane XOY is rhombic contains the translations $\tau_x+\tau_z$ and $\tau_y+\tau_z$ †, then the nets in the planes XOZ, YOZ are also rhombic, and a primitive triplet of the lattice is

$$2\tau_1=\tau_y+\tau_z,\ 2\tau_2=\tau_z+\tau_x,\ 2\tau_3=\tau_x+\tau_y \quad . \quad . \quad (3).$$

This lattice (Γ_o'') may be formed by putting points at the vertices and at the centres of all the faces of a series of rectangular parallelepipeda (of which OA, OB, OC are concurrent edges).

If no one of the nets in the planes YOZ, ZOX, XOY is rhombic the lattice cannot contain $\tau_y+\tau_z$, $\tau_z+\tau_x$, or $\tau_x+\tau_y$. If it does not contain $\tau_x+\tau_y+\tau_z$, we have Γ_o; if it does, we have a lattice (Γ_o''') such that

$$\tau_x+\tau_y+\tau_z \text{ and any two of } 2\tau_x,\ 2\tau_y,\ 2\tau_z \quad . \quad . \quad (4)$$

form a primitive triplet.

Other primitive triplets are

$$2\tau_1=\tau_y+\tau_z-\tau_x,\ 2\tau_2=\tau_z+\tau_x-\tau_y,\ 2\tau_3=\tau_x+\tau_y-\tau_z \quad (5),$$

$$2\tau_1=\tau_x+\tau_y+\tau_z,\ 2\tau_2=\tau_x-\tau_y+\tau_z,\ 2\tau_3=2\tau_z \quad . \quad (6).$$

This lattice may be formed by placing points at the vertices and at the centres of the same series of rectangular parallelepipeda as were used to form Γ_o''.

The symmetry of all the lattices of this section is evidently that of D_{2h}, for they each have three 2-al axes and a centre of symmetry.

* This figure applies to lattices of this type if OA is supposed to be perpendicular to OB, and if we take $OA=2\tau_x$, $OB=2\tau_y$, $OC=2\tau_z$.

† If it contains one it contains the other, for

$$(\tau_x+\tau_z)+(\tau_y+\tau_z)=(\tau_x+\tau_y)+2\tau_z.$$

Comparing the triplet (1) of this section with (1) of § 6, we see that Γ_o is a particular (or 'specialized') case of Γ_m; similarly comparing (3) and (4) with (5) and (6) of § 6, we see that Γ_o'' and Γ_o''' are specialized forms of Γ_m'; and comparing (2) with (1) or (4) of § 6, we see that Γ_o' may be regarded as a specialized form of either Γ_m or Γ_m'.

§ 8. Suppose now that a lattice has a 4-al rotation-axis OZ. Then it has a square net perpendicular to OZ (p. 130); let $2\tau_x$, $2\tau_y$ be the smallest translations of the group represented by the lattice in the plane of this net. Then $2\tau_x$, $2\tau_y$ are a primitive pair, and are at right angles and equal to one another. Let the components of any other translation of the group parallel and perpendicular to OZ be τ and τ'. Then since OZ is a 4-al axis of the lattice there is a translation of the group whose components parallel and perpendicular to OZ are τ and $-\tau'$. Therefore $2\tau = (\tau+\tau')+(\tau-\tau')$ and $2\tau' = (\tau+\tau')-(\tau-\tau')$ are translations of the group, so that τ is a multiple of τ_z, and τ' is of the form $m\tau_x+p\tau_y$. Hence any translation of the group is of the form $m\tau_x+n\tau_y+p\tau_z$.

$m\tau_x+n\tau_y+p\tau_z-2n\tau_y-2p\tau_z = m\tau_x-n\tau_y-p\tau_z$ is also a translation of the group, so that the direction of τ_x is a 2-al rotation-axis of the lattice; similarly the direction of τ_y is a 2-al axis.

Hence lattices of this type have three mutually orthogonal rotation-axes and are therefore specialized forms of those of § 7.

Γ_o, which is formed by placing points at the vertices of right prisms with rectangular bases, becomes, if OZ is a 4-al axis, a lattice formed by placing points at the vertices of right prisms with *square* bases (such as *L'N'G'P' LNGP*, *G'D'F'E' GDFE* of Fig. 73 *).

The figure shows that the lattice may also be formed by placing points at the vertices and at the centres of the bases of right prisms with square bases (such as *N'D'E'P' NDEP*, *D'S'U'E' DSUE*), proving that if OZ is a 4-al axis Γ_o and Γ_o' are identical.

A primitive triplet of this lattice Γ_t is

$$2\tau_1 = 2\tau_x,\ 2\tau_2 = 2\tau_y,\ 2\tau_3 = 2\tau_z \quad . \quad . \quad . \quad (1),$$

being the same as for Γ_o †.

Γ_o'' becomes a lattice formed by placing points at the

* The points such as h, k, do not belong to the lattice in this case.

† Γ_o and Γ_t are distinguished by the fact that in the latter τ_x and τ_y are equal in magnitude.

vertices and at the centres of all the faces of right prisms with square bases (such as $N'D'E'P'$ $NDEP$, $D'S'U'E'$ $DSUE$*). The figure shows that this lattice may also be formed by placing points at the vertices and at the centres of right prisms with square bases (such as $L'N'G'P'$ $LNGP$, $G'D'F'E'$ $GDFE$), proving that if OZ is a 4-al axis Γ_o'' and Γ_o''' are identical.

A primitive triplet of this lattice Γ_t' is

$$2\tau_1 = \tau_y + \tau_z,\ 2\tau_2 = \tau_z + \tau_x,\ 2\tau_3 = \tau_x + \tau_y \quad . \quad . \quad (2),$$

being the same as for Γ_o'''.

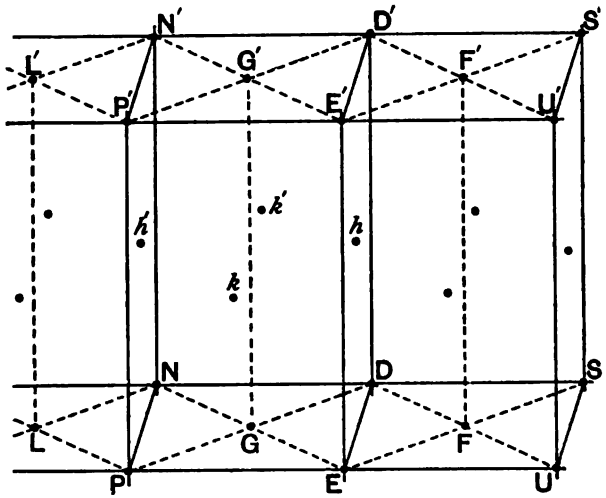

Fig. 78.

These lattices have the symmetry of the group D_{4h}, for they have a 4-al and a perpendicular 2-al rotation-axis passing through a centre of symmetry.

§ 9. Now suppose that the lattice has three mutually orthogonal 2-al axes as in § 7, but that τ_x, τ_y, τ_z are all equal in magnitude. Γ_o may then be considered as formed by placing points at the vertices of a series of cubes. We denote the lattice in this case by Γ_r; a primitive triplet is

$$2\tau_1 = 2\tau_x,\ 2\tau_2 = 2\tau_y,\ 2\tau_3 = 2\tau_z \quad . \quad . \quad . \quad . \quad (1).$$

Γ_o', which is formed by placing points at the vertices and at the centres of *one* pair of faces of rectangular parallelepipeda,

* Now the points h, k, &c., belong to the lattice.

can have no symmetry when τ_x, τ_y, τ_z are equal, which it does not have when τ_x is equal to τ_y but not to τ_z.

Γ_o'' is now constructed by placing points at the vertices and at the centres of all the faces of a series of cubes, and is denoted by Γ_r'; a primitive triplet is

$$2\tau_1 = \tau_y + \tau_z,\ 2\tau_2 = \tau_z + \tau_x,\ 2\tau_3 = \tau_x + \tau_y \quad . \quad . \quad (2).$$

Γ_o''' is now constructed by placing points at the vertices and at the centres of a series of cubes, and is denoted by Γ_r''; a primitive triplet is

$$2\tau_1 = \tau_y + \tau_z - \tau_x,\ 2\tau_2 = \tau_z + \tau_x - \tau_y,\ 2\tau_3 = \tau_x + \tau_y - \tau_z \quad (3).$$

Γ_o'' and Γ_o''' are not in this case identical, for if *N'D'E'P' NDEP*, *D'S'U'E' DSUE* are cubes (Fig. 73), *L'N'G'P' LNGP*, *G'D'F'E' GDFE* are not, and vice versa.

Γ_r, Γ_r', and Γ_r'' have evidently *OX*, *OY*, *OZ* as 4-al axes; their symmetry is that of O_h, for they each have three 4-al rotation-axes passing through a centre of symmetry.

§ 10. Suppose now that a lattice has a 3-al axis, and let $OC = 2\tau_z$ be the primitive translation in its direction of the translation-group represented by the lattice.

First suppose that the component parallel to *OC* of every translation of the group is zero or a multiple of $2\tau_z$. Let $2\tau'$, $2\tau''$ be two translations which with $2\tau_z$ form a primitive triplet; and let their components parallel to *OC* be $2m\tau_z$, $2n\tau_z$ respectively. Then $OE = 2\tau' - 2m\tau_z$, $OF = 2\tau'' - 2n\tau_z$ form with $2\tau_z$ a primitive triplet (p. 122) and are perpendicular to *OC*. The net perpendicular to *OC* must be equilateral, since it is brought to self-coincidence by a rotation through $\frac{2\pi}{3}$ about *OC*. Let $OA = 2\tau_1$, $OB = 2\tau_2$ be a primitive pair of this net whose directions make an angle of $\frac{2\pi}{3}$ with one another. Then since the triangles *OAB*, *OEF* are equal, therefore the tetrahedra *OABC*, *OEFC* are equal, and hence $2\tau_1$, $2\tau_2$, $2\tau_z$ are a primitive triplet of the lattice. The lattice (Γ_h) may, in this case, be formed by placing points at the vertices of a series of right prisms (of height *OC*) whose bases are rhombic with angles $\frac{\pi}{3}$ and $\frac{2\pi}{3}$.

This arrangement is shown in Fig. 74; an inspection of the figure shows that *OC* is not only a 3-al but is also a 6-al rotation-axis, and that *OA* is a 2-al rotation-axis of the lattice.

Hence Γ_h has the symmetry of D_{6h}; for it possesses a 6-al rotation-axis and a perpendicular 2-al axis passing through a centre of symmetry.

Now suppose that the translation-group represented by the lattice contains a translation whose projection on OC is not zero or a multiple of OC. Combining this translation with multiples of $2\tau_z$ we see that there must be a translation whose projection on OC (or OC produced) is $\geqslant \frac{1}{2}OC$.

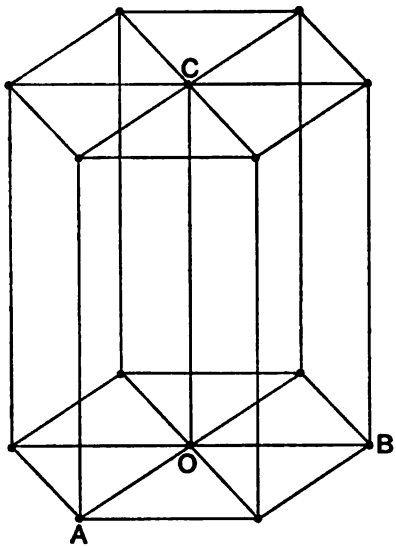

Fig. 74.

Let OA_1 represent that translation whose projection on OC (or OC produced) is least*. First suppose that OA_1 makes an acute angle with OC, and let A_1 be brought to A_2, A_3 by rotation through $\frac{2\pi}{3}$, $\frac{4\pi}{3}$ about OC.

Then $OA_1 = 2\tau'$, $OA_2 = 2\tau''$, $OA_3 = 2\tau'''$ are all translations of the group represented by the lattice, and therefore

$$2\tau' + 2\tau'' + 2\tau'''$$

is a translation of the group. It is evidently parallel to OC, and is therefore a multiple of $2\tau_z$; for $2\tau_z$ is the primitive translation in this direction. But the projections of OA_1, OA_2, OA_3 on OC are all $\geqslant \tau_z$; therefore

$$2\tau' + 2\tau'' + 2\tau''' \text{ is } \geqslant 3\tau_z$$

and is a multiple of $2\tau_z$, and hence $2\tau' + 2\tau'' + 2\tau''' = 2\tau_z$.

Similarly if OA_1 makes an obtuse angle with OC,

$$2\tau' + 2\tau'' + 2\tau''' = -2\tau_z;$$

and the following reasoning holds good if we substitute C' for C, where $OC' = -2\tau_z$.

Now of all the translations whose projections on OC (or OC produced) are equal to that of OA_1, choose OA_1 to be the one which makes with OC (or OC produced) an acute angle as small as possible.

* Excluding translations whose component parallel to OC is zero.

Draw the parallelepipedon $OCA_1A_2A_3A_1'A_2'A_3'$ (Fig. 75); then, if Q be any point of the lattice, we see, by combining the translation OQ with multiples of $2\tau'$, $2\tau''$, $2\tau'''$, that there must be a point P of the lattice inside this parallelepipedon or upon its surface. Now, if P does not coincide with O or C, it must lie in one of the planes $A_1A_2A_3$, $A_1'A_2'A_3'$, for otherwise OP, A_1P, or CP would represent a translation whose projection on OC is less than that of OA_1. This being so P must coincide with A_1, A_2, A_3, A_1', A_2', or A_3', for otherwise either OP or CP would make an angle with $OC < COA_1$.

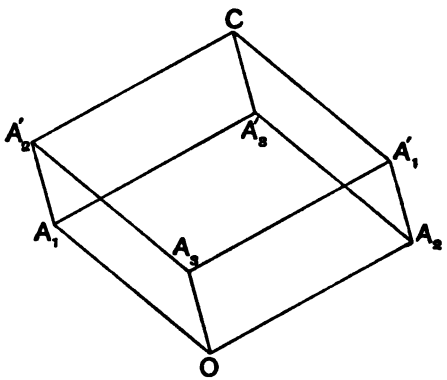

Fig. 75.

Hence $2\tau'$, $2\tau''$, $2\tau'''$ are a primitive triplet of the lattice (Γ_{rh}); which may be formed by placing points at the vertices of regular rhombohedra such as that of Fig. 75.

The plane COA_1 is evidently a plane of symmetry for this lattice; it possesses a centre of symmetry lying on a 3-al rotation-axis, and a symmetry-plane through this axis, and has therefore the symmetry of D_{3d}*.

§ 11. We have now proved that there are fourteen different kinds of lattice; it should be noticed that they all possess symmetry belonging to some *holohedral* class of crystals.

There is 1	whose symmetry is that of the holohedry of the	triclinic system.
There are 2	" " "	monoclinic system.
There are 4	" " "	orthorhombic system.
There are 2	" " "	tetragonal system.
There is 1	" " "	rhombohedral system.
There is 1	" " "	hexagonal system.
There are 3	" " "	regular system.

* If $\cos COA_1 = \sqrt{\frac{1}{2}}$, $\sqrt{\frac{2}{3}}$, or $\frac{1}{2}$, Γ_{rh} becomes respectively Γ^r, Γ_r', or Γ_r''.

CHAPTER XV

THE BRAVAIS STRUCTURE-THEORY.

§ 1. The theory of crystal-structure put forward by Bravais in 1850 marks a very important stage in the historical development of the subject.

Bravais considers all molecules of a crystalline material to be alike and similarly orientated, and their centres of mass to be placed at the points of one of the fourteen possible lattices which we discussed in the previous chapter.

The individual molecules are assumed to themselves possess symmetry, and this symmetry is supposed to influence the symmetry of the medium as a whole.

Let O, A, B be any three points of a lattice, and let ω, α, β be the molecules whose centres of mass are at these three points.

Suppose l any axis of the first or second sort, then l evidently cannot be a symmetry-element of the medium as a whole unless it is an element of the lattice. Let l be a symmetry-element of the lattice, and let the corresponding operation leave O unmoved, bring A to B, ω to ω', and α to β'.

Then since ω, α are similarly orientated, so are ω', β'.

Now if l is a symmetry-element of ω (as well as an element of the lattice), ω' coincides with ω, and therefore β' coincides with β. Hence a symmetry-element of the lattice and of one of the molecules is a symmetry-element of the structure as a whole; and hence is derived the statement that *the symmetry-elements of a crystal are those common both to the lattice and to one of the individual molecules.*

If we had supposed that the symmetry of the individual molecules had no effect on the symmetry of the crystal as a whole, or if we had supposed the molecules to be always spherical, we could only account for holohedral crystals; but by supposing that the molecules do affect the symmetry, and that the molecules can have the same symmetry as merohedral crystals, we can explain the existence of a crystal belonging to any one of the thirty-two classes, in spite of the fact that all lattices are of holohedral symmetry.

Theoretically we might have molecules of a low order of symmetry in a lattice of high order (thus we might have molecules of the symmetry of D_2 in a lattice of the regular system; the crystal would then have the symmetry of D_2 if the rotation-axes of any one of the molecules were parallel to the three 4-al axes of the lattice, the symmetry of C_2 if only one of the axes of the molecule was parallel to a 4-al or 2-al axis of the lattice, and the symmetry of C_1 if no axis of the molecule was parallel to a 2-al or 4-al axis of the lattice: or again we might have molecules of the symmetry of D_{2h} in the lattice Γ_{rh}, the crystal could then have at most the symmetry of C_{2h}). Experience shows, however, that this does not often happen—crystals do not in general possess faces whose normals make the same angles with one another as in the case of crystals of higher symmetry—and there are also (according to Bravais) theoretical reasons why molecules should crystallize in a lattice whose symmetry is as like their own as possible (thus molecules, whose symmetry is that of C_3, C_{3i}, C_{3v}, D_3 or D_{3d}, usually crystallize in the lattice Γ_{rh} with their symmetry-elements parallel to the corresponding elements of the lattice; and the symmetry of the resulting crystal is the same as that of the individual molecules). An obvious objection to Bravais' theory is that the difficulty of explaining crystal-structure is only deferred (reculée), for we are left with the difficulty of explaining the symmetry of the individual molecules; Bravais' reply that 'the atomic theory furnishes us with an answer ready to hand, for it represents each molecule of a body as composed of a finite number of atoms of different sorts,' can hardly be considered satisfactory.

§ 2. It has been seen that in most systems more than one lattice is possible; Bravais tries to allot suitable lattices to certain crystals by means of the following considerations.

Firstly, cleavage surfaces are probably parallel to net-planes of the lattice in which points are most closely packed, and which consequently belong to sets whose 'interval' is comparatively large (p. 115).

Secondly, he gives reasons for assuming that the faces of most frequent occurrence on a crystal are also parallel to similar net-planes*.

* 'Car les mouvements internes qui agissent incessamment sur la superficie de la masse cristalline en voie de formation jouent, jusqu'à un certain point, le rôle des forces extérieures qui s'emploient à cliver un crystal; ils ont dû, par conséquent, respecter de préférence les séries moléculaires qui se trouvent disposées suivant des plans réticulaires dont les réseaux ont les tissus les plus serrés.'

Then he calculates the density of points in various net-planes of a lattice, and deduces the most probable cleavage and the forms of most frequent occurrence in crystals whose molecules are arranged at the points of such a lattice. By comparing his results with the cleavage and form of crystals found in nature he is able to assign to various crystals their appropriate lattices.

It is evident that the 'law of rational indices' is a direct conclusion from Bravais' theory that net-planes whose points are most closely packed are those parallel to the faces of most frequent occurrence, for it is exactly such faces whose indices can readily be expressed as comparatively small integers.

Again, we deduce from the theory that every symmetry-plane of a crystal is parallel to a possible face and that the line perpendicular to it is parallel to a possible edge; also that every symmetry-axis is parallel to a possible edge and that the plane perpendicular to it is parallel to a possible face *. We deduce moreover that every symmetry-axis of a crystal is 2-al, 3-al, 4-al, or 6-al †. These deductions are in accordance with observation.

§ 3. Bravais' explanation of twinning is interesting. In those cases where two crystals belonging to a merohedral class are twinned in such a way that one is derived from the other by an operation (rotation about a 2-al axis, or reflexion in a plane) which is a symmetry-operation of the corresponding holohedral class ‡, he supposes that the two crystals have the *same* lattice, but that the molecules in the two members of the twin have different orientations; such twins are more or less inter-penetrant. In those cases where one member cannot be derived from the other by rotation about a 2-al axis, but only by reflexion in a plane the molecules of the two crystals must be enantiomorphous. If one member of a twin is derived from the other by rotation about a 2-al axis l which is not a symmetry-axis of the lattice, Bravais supposes that the two crystals possess similar but differently orientated lattices, one lattice being derived from the other by the same operation as in the case of the crystals themselves. The orientation of the molecules relatively to the lattice is the same in each crystal. The members of the twin do not inter-penetrate but are united along a net-plane perpendicular to l. This net belongs to both crystals for it is brought into self-coincidence

* pp. 181, 182. The cases of a 3-al axis of the first sort and a 6-al axis of the second sort are no longer exceptional.

† p. 182.

‡ This is a case often met with; it is called 'supplementary twinning.'

by a rotation through π about l. Bravais admits that there are some cases of twinning which are not readily explained on his theory.

§ 4. Bravais' structure-theory is interesting historically, for it has had very considerable influence on subsequent developments of the subject. It is not at the present day considered quite satisfactory, since, in addition to objections already mentioned, it does not include all possible or probable arrangements of molecules or atoms which may be suggested as representing the structure of various crystals. In particular, as Sohncke pointed out, an arrangement of molecules similar to that of the vertices of the cells in a honeycomb is not provided for in Bravais' theory; but such an arrangement is simple and might well represent the structure of certain crystals. Before we can discuss other theories we must investigate the number and properties of infinite groups of movements; this investigation will occupy chapters xvi to xxiii.

CHAPTER XVI

PROPERTIES OF GEOMETRICAL OPERATIONS.

§ 1. We proved in chapter iii that any operation which can bring a figure into coincidence with a congruent or enantiomorphous figure is equivalent to a translation, rotation, rotatory-reflexion, or to a combination of a translation and a rotation. Since the 'product' of a series of operations of this kind must also be an operation bringing the figure into coincidence with a congruent or enantiomorphous figure, the product is itself a rotatory-reflexion, a translation, a rotation, or a combination of the last two. We shall proceed to find the result of combining various operations in some useful cases.

§ 2. *A rotation $A(\alpha)$ through an angle α about an axis a, and a translation T parallel to a are permutable operations.*

For let P be any point, and let P be brought to P_2 by $A(\alpha)$; let P, P_2 be brought to P_1, P_3 by T. Then evidently P is brought to P_1 and then to P_3 by T followed by $A(\alpha)$, and to P_2 and then to P_3 by $A(\alpha)$ followed by T; but P is *any* point, and $\therefore T.A(\alpha) = A(\alpha).T$.

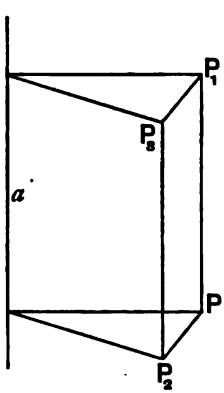

Fig. 76.

The combination of a rotation about an axis and a translation parallel to it is called *a screw* about that axis; and if such a combination brings a figure U to self-coincidence the axis is called a *screw-axis of symmetry* for U, or more briefly a *screw-axis* of U. If the angle of rotation about the axis (a) is α, and the length of the translation is t, the screw about a is usually represented by the symbol $A_{\alpha, t}$.

This screw will be called 'a screw of angle α and translation t'; or, if $n\alpha = 2\pi$, 'an n-al screw of translation t.'

A screw is said to be right-handed or left-handed according as, when we look along the axis in the direction of the translation, the rotation appears to be in the positive (clockwise) or negative direction.

§ 3. *Any translation T, followed by a rotation $A(\omega)$ about an axis a, perpendicular to the translation, is equivalent to a rotation through the same angle about a parallel axis.*

Let A_1 be any point on a; let C be a point such that CA_1 represents the translation. Take a point B_1 such that the angle $B_1CA_1 = CA_1B_1 = \frac{\pi-\omega}{2}$, and draw B_1D parallel and equal to CA_1. Then CA_1DB_1 is a parallelogram and the angle $A_1B_1C = B_1A_1D = \omega$.

Now T brings the points C, B_1 to A_1, D respectively, and $A(\omega)$ brings A_1, D to A_1, B_1 respectively. Therefore $T.A(\omega)$ brings the points C, B_1 to A_1, B_1 respectively.

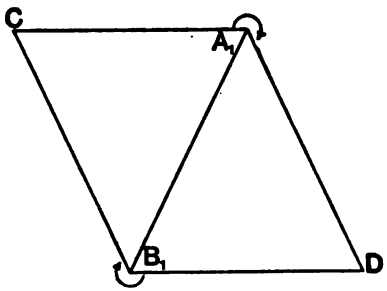

Fig. 77.

Now a rotation $B(\omega)$ about an axis b through B_1 parallel to a does the same; therefore $T.A(\omega)$ and $B(\omega)$ bring three non-collinear points (viz. C, and any two points on a) into the same positions, and therefore $T.A(\omega)$ brings any figure into the same position as $B(\omega)$ does; hence $T.A(\omega) = B(\omega)$.

Again, since $T.A(\omega) = B(\omega)$, therefore

$$B(-\omega).T.A(\omega) = 1 \text{ (for } \{B(\omega)\}^{-1} = B(-\omega)\text{), and}$$
$$B(-\omega).T = A(-\omega), \text{ or } B(\phi).T = A(\phi) \text{ (putting } \phi \text{ for } -\omega).$$

Hence, *a rotation about any axis, followed by a translation perpendicular to that axis, is equivalent to a rotation through the same angle about a parallel axis.*

Again, since $T.A(\omega) = B(\omega)$, therefore $B(\omega).A(-\omega) = T$.

Hence, *a rotation about any axis, followed by a rotation through the same angle in the opposite direction about a parallel axis, is equivalent to a translation perpendicular to the axes.*

This is also evident from Euler's construction (p. 24) for

a rotation through an infinitesimal angle about an infinitely distant axis is equivalent to a translation perpendicular to that axis.

From Fig. 77 we see that $CA_1 = 2A_1B_1 \sin\frac{\omega}{2}$; we deduce from this that if $\omega = \pi$, $CA_1 = 2A_1B_1$ and the angle $CA_1B_1 = 0$; that is B_1 bisects the line CA_1. Hence a translation $A_{-1}A_1$, followed by a rotation through π about an axis through A_1 (see Fig. 64), is equivalent to a rotation through π about a parallel axis through O; a rotation through π about an axis through A_{-1}, followed by a similar rotation about a parallel axis through O, is equivalent to the translation $A_{-1}A_1$, &c.

If $\omega = \frac{2\pi}{3}$, $CA_1 = \sqrt{3}.A_1B_1$ and $CA_1B_1 = \frac{\pi}{6}$; that is, B_1 is at the centre of an equilateral triangle CA_1E whose side is CA_1 (Fig. 78). We have

$$CA_1.A\left(\frac{2\pi}{3}\right) = B\left(\frac{2\pi}{3}\right),$$

$$B\left(\frac{2\pi}{3}\right).EA_1 = A\left(\frac{2\pi}{3}\right),$$

$$A\left(\frac{2\pi}{3}\right).B\left(\frac{-2\pi}{3}\right) = A_1C,$$

$$A\left(\frac{-2\pi}{3}\right).B\left(\frac{2\pi}{3}\right) = A_1E, \text{ &c., &c.}$$

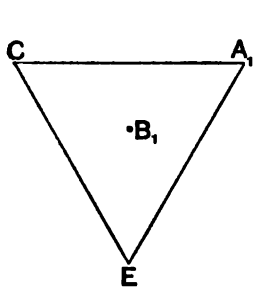

Fig. 78.

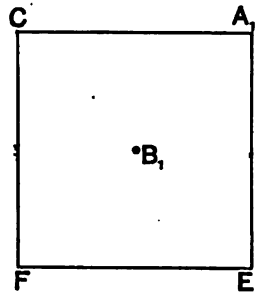

Fig. 79.

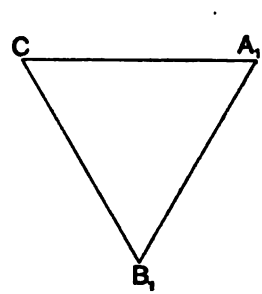

Fig. 80.

If $\omega = \frac{\pi}{2}$, $CA_1 = \sqrt{2}.A_1B_1$ and $CA_1B_1 = \frac{\pi}{4}$; that is, B_1 is at the centre of a square CA_1EF whose side is CA_1 (Fig. 79).

We have

$$CA_1 . A\left(\frac{\pi}{2}\right) = B\left(\frac{\pi}{2}\right),$$

$$B\left(\frac{\pi}{2}\right) . EA_1 = A\left(\frac{\pi}{2}\right),$$

$$A\left(\frac{\pi}{2}\right) . B\left(\frac{-\pi}{2}\right) = A_1 C,$$

$$A\left(\frac{-\pi}{2}\right) . B\left(\frac{\pi}{2}\right) = A_1 E, \text{ \&c., \&c.}$$

If $\omega = \frac{\pi}{3}$, $CA_1 = A_1 B_1$ and $CA_1 B_1 = \frac{\pi}{3}$; that is, B_1 is at the third vertex of an equilateral triangle whose side is CA_1 (Fig. 80). We have

$$CA_1 . A\left(\frac{\pi}{3}\right) = B\left(\frac{\pi}{3}\right),$$

$$B\left(\frac{\pi}{3}\right) . B_1 C = A\left(\frac{\pi}{3}\right),$$

$$A\left(\frac{\pi}{3}\right) . B\left(\frac{-\pi}{3}\right) = A_1 C,$$

$$A\left(\frac{-\pi}{3}\right) . B\left(\frac{\pi}{3}\right) = CB_1, \text{ \&c., \&c.}$$

§ 4. *Any translation T, followed by a rotation $A(\omega)$ about an axis a, is equivalent to a screw $B_{\omega, t}$ about some axis parallel to a, where t is the component of T in the direction of a.*

For let T be resolved into components t, t' parallel and perpendicular to a. Then $T . A(\omega)$ is equivalent to a translation t parallel to a, followed by a translation t' perpendicular to a, and a rotation through ω about a; i. e. to a translation t parallel to a, followed by a rotation through ω about some axis b parallel to a (by § 3). Therefore $T . A(\omega) = B_{\omega, t}$. Similarly we can prove $A(\omega) . T = B'_{\omega, t}$.

§ 5. We can now state the result quoted at the beginning of this chapter in the form:—

'*Every operation which brings a figure into coincidence with a congruent figure is equivalent to a screw, and every operation which brings a figure into coincidence with an enantiomorphous figure is equivalent to a rotatory-reflexion.*'

A screw is said to be an operation of the first sort, and a rotatory-reflexion an operation of the second sort.

The product of any number of screws and rotatory-reflexions is equivalent to a screw or rotatory-reflexion, according as the number of rotatory-reflexions in the product is even or odd *.

The following particular cases of the screw $A_{\omega, t}$ are interesting:—

(1) $\omega = 0$; the screw becomes a translation.
(2) $t = 0$; the screw becomes a rotation.

The following particular cases of a rotatory-reflexion are interesting:—

(1) Angle of rotation $= 0$; the rotatory-reflexion becomes a simple reflexion.

(2) Angle of rotation $= \pi$; the rotatory-reflexion becomes an inversion about the intersection of the axis and plane of the rotatory-reflexion.

(3) Angle of rotation infinitesimal and axis at an infinite distance; the rotatory-reflexion becomes a reflexion in a plane combined with a translation parallel to that plane †.

The plane is called, in this case, a *gliding-plane* and the operation a *gliding-reflexion*. A gliding-reflexion whose gliding-plane is σ, and whose translation (parallel to σ) is represented by t is usually denoted by the symbol $S(t)$.

§ 6. We now proceed to find the result of combining various operations in some useful cases.

First take the case of a screw $A_{\omega, t}$ about an axis a, and any translation T; let the components of T parallel and perpendicular to a be t' and t''. Then $A_{\omega, t} . T$ is equivalent to a rotation through ω about a, followed by a translation t parallel to a, another translation t' parallel to a, and a translation t'' perpendicular to a; that is, to a rotation through ω about a, followed by a translation t'' perpendicular to a, and a translation $t + t'$ parallel to a (for all translations are permutable operations ‡); that is, to a rotation through ω about an axis b parallel to a, followed by a translation $t + t'$ parallel to a or b; that is, to a screw $B_{\omega, t+t'}$ about b.

Similarly we may prove $T . A_{\omega, t} = B'_{\omega, t+t'}$ about some other axis b' parallel to a.

* p. 80.

† Since rotation about an axis and reflexion in a perpendicular plane are permutable (p. 31), proceeding to the limit when the axis is at infinity and the angle of rotation infinitesimal, we see that a reflexion in a plane and a translation parallel to that plane are permutable operations. This may easily be proved independently.

‡ See p. 118.

Now we shall take the case of two screws A_{α, t_1}, B_{β, t_2} about two parallel axes a, b.

Then $A_{\alpha, t_1} . B_{\beta, t_2}$ = a translation t_1 parallel to a, followed by rotations $A(\alpha)$, $B(\beta)$*, and a translation t_2 parallel to b (or a). But $A(\alpha) . B(\beta)$ is equivalent to a rotation $C(\alpha+\beta)$ about some axis c parallel to a and b (p. 24); therefore $A_{\alpha, t_1} . B_{\beta, t_2}$ is equivalent to the screw $C_{\alpha+\beta, t_1+t_2}$, for $C(\alpha+\beta)$ is permutable with translations parallel to c (p. 146).

Putting $\alpha+\beta = 0$, we see that the product of screws A_{α, t_1}, $B_{-\alpha, t_2}$ about two parallel axes a and b = a translation whose component parallel to a or b is t_1+t_2.

§ 7. Let $C_{\gamma, t}$ and $C'_{\gamma, 0}$ be a screw and a rotation about two parallel axes. Then since $C_{\gamma, t} . C'_{-\gamma, 0}$ = some translation T,

$$\therefore \quad C_{\gamma, t} = T . (C'_{-\gamma, 0})^{-1} = T . C'_{\gamma, 0}.$$

Again, since $C'_{-\gamma, 0} . C_{\gamma, t}$ = some translation T',

$$\therefore \quad C_{\gamma, t} = C'_{\gamma, 0} . T' = T . C'_{\gamma, 0}.$$

Conversely if $C'_{\gamma, 0}$ is any rotation and T any translation, a translation T' can be found such that $T . C'_{\gamma, 0} = C'_{\gamma, 0} . T'$.

§ 8. We proceed now to the more general case of any two screws A_{α, t_1}, B_{β, t_2} about any two axes a and b.

Take *any* point P and draw lines a', b' through P parallel to a and b. Let $C'_{\gamma, 0}$ be the rotation about an axis through P which is equivalent to $A'(\alpha) . B'(\beta)$.

Then by § 7 $A_{\alpha, t_1} . B_{\alpha, t_2} = T_1 . A'_{\alpha, 0} . B'_{\beta, 0} . T_2 = T_1 . C'_{\gamma, 0} . T_2$ $= C'_{\gamma, 0} T' . T_2 = C'_{\gamma, 0} . T =$ (similarly) $T . C'_{\gamma, 0}$, where T_1, T_2, T', T, T are certain translations.

Repetition of this process shows that the product of any screws $A_{\alpha, t_1} . B_{\beta, t_2} . C_{\gamma, t_3} . D_{\delta, t_4} \ldots$ about axes a, b, c, d, ... $= M'_{\mu, 0}$ combined with some translation, where $M'_{\mu, 0}$ is the result of combining rotations through α, β, γ, δ, ... about axes through *any* point respectively parallel to a, b, c, d, ...

The practical method of finding the resultant of any number of screws is as follows: take a point Q which is brought by $A_{\alpha, t_1} . B_{\beta, t_2} . C_{\gamma, t_3} \ldots$ into an easily ascertained position P. Find, by Euler's construction, the position of the axis (m') and the angle of rotation (μ) of the resultant of rotations through α, β, γ, ... about axes through P parallel to a, b, c, ... respectively. Then $A_{\alpha, t_1} . B_{\beta, t_2} . C_{\gamma, t_3} \ldots$ is equivalent to some

* These can also be written $A_{\alpha, 0}$, $B_{\beta, 0}$.

translation T followed by the rotation $M'_{\mu, 0}$ about m', and therefore $T.M'_{\mu, 0}$ brings Q to P.

Hence (since P lies on m') T is the translation QP. We have now found completely the operations T and $M'_{\mu, 0}$. Let t and t' be the components of T parallel and perpendicular to m'. Find by the construction of p. 147 the position of the axis m such that $M_{\mu, 0}$ = the translation t' followed by the rotation $M'_{\mu, 0}$.

Then $A_{\alpha, t_1}.B_{\beta, t_2}.C_{\gamma, t_3}.\ldots$ = the screw $M_{\mu, t}$ about the axis m.

A useful example is the following:—

'Find the resultant of a screw $A_{\pi, 2t_a}$ about an axis a, followed by a screw $B_{\pi, 2t_b}$ about an axis b, if a is at a distance t_c from b and makes an angle θ with it.'

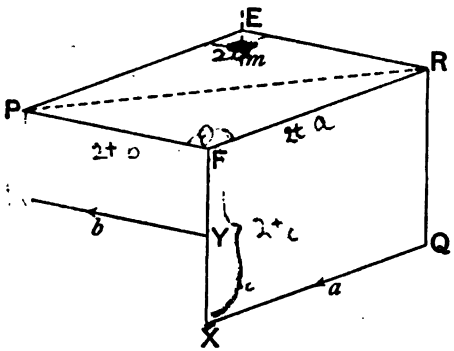

Fig. 81.

Let the line perpendicular to a and b meet them in X and Y; then $XY = t_c$. Produce XY to F making $YF = XY$, and draw FR, FP parallel to a and b making

$$RF = 2t_a, \quad FP = 2t_b.$$

Draw RQ parallel to YX meeting a in Q; then $QX = 2t_a$. Make the angles RPE, ERP in the plane FPR each $= \frac{\pi}{2} - \theta$. (Fig. 81).

Then $A_{\pi, 2t_a}$ brings Q to X, and $B_{\pi, 2t_b}$ brings X to P.

$$\therefore \quad A_{\pi, 2t_a}.B_{\pi, 2t_b} \text{ brings } Q \text{ to } P.$$

Now rotations through π, first about an axis through P parallel to a, and then about an axis through P parallel to b, are equivalent to a rotation through 2θ about an axis through P parallel to XY (p. 25). Hence $A_{\pi, 2t_a}.B_{\pi, 2t_b}$ = the translation QP (that is, a translation QR and a translation RP), followed by a rotation through 2θ about an axis through P parallel to XY = the translation $QR\,(= 2t_c)$ followed by a rotation $M_{2\theta, 0}$ about an axis m through E parallel to XY (p. 147) $= M_{2\theta, 2t_c}$.

The distance of m from a = the distance of E from FR

$$= ER.\sin\left(\frac{\pi}{2} - \theta + PRF\right) = ER.\cos(\theta - PRF)$$

$$= \frac{PR.\cos(\theta - PRF)}{2\sin\theta} = \frac{PR.\cos RPF}{2\sin\theta}$$

$$= \frac{PF - FR.\cos RFP}{2\sin\theta} = \frac{t_b + t_a\cos\theta}{\sin\theta}.$$

Similarly the distance of m from $b = \dfrac{t_a + t_b\cos\theta}{\sin\theta}$.

The following special cases are of importance:—

(1) $\theta = \dfrac{\pi}{2}$.

'The product of three screws of angle π whose axes are three perpendicular sides (of lengths t_a, t_b, t_c) of a rectangular

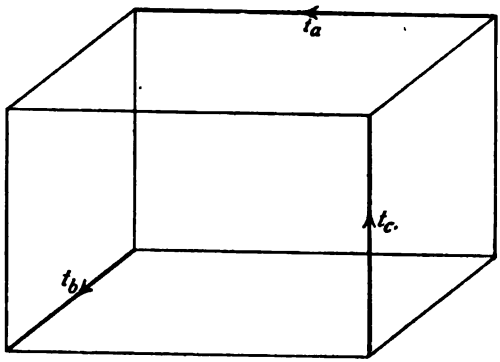

Fig. 82.

parallelepiped which do not cut, and whose translations are twice the respective sides taken in order *, is unity.'

(2) $t_a = t_b = 0$.

'Two rotations through π about axes making an angle θ with one another, and at a distance t_c apart, are equivalent to a screw of angle 2θ and translation $2t_c$ about the line meeting and perpendicular to both axes.'

The case of $\theta = \dfrac{\pi}{2}$ is of especial importance.

(3) $t_c = 0$, $\theta = \dfrac{\pi}{2}$.

* That is, taken in directions such that $t_a + t_b + t_c$ = a translation represented by that diagonal of the parallelepiped which meets none of the three axes. Figure 82 makes this clear.

'Two screws of angle π and translations $2t_a$ and $2t_b$ about perpendicular intersecting axes a and b are equivalent to a rotation through π about an axis perpendicular to a and b, and at distances t_b, t_a from a, b respectively.'

(4) Case (2) may be put in the following form:—

'A rotation through π about an axis a, followed by a screw of angle 2θ and translation $2t_c$ about a line c perpendicular to a and meeting it, is equivalent to a rotation through π about an axis perpendicular to and meeting c, and making an angle θ with and at a distance t_c from a.'

§ 9. We shall now consider operations of the second sort.

Suppose we have reflexions S, S_1 in two parallel planes σ, σ_1; *then $S.S_1$ is equivalent to a translation perpendicular to σ and σ_1 through a distance equal to twice that between σ and σ_1.*

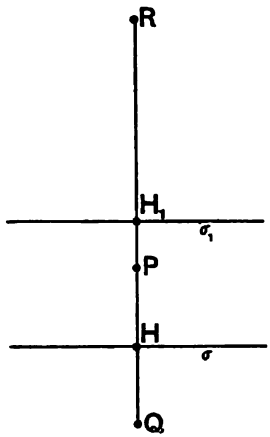

Fig. 83.

For let P be *any* point, Q its reflexion in σ, and R the reflexion of Q in σ_1. Let RPQ meet σ, σ_1 in H and H_1 (Fig. 83).

Then

$$\begin{aligned} PR &= PH_1 + H_1R = PH_1 + QH_1 \\ &= 2PH_1 + QP = 2PH_1 + 2HP \\ &= 2HH_1; \end{aligned}$$

so that any point P is translated by $S.S_1$ through a distance $2HH_1$ perpendicular to σ and σ_1.

It will be seen that the above is a limiting case of the theorem of p. 26.

§ 10. There is now no difficulty in combining any number of screws and rotatory-reflexions.

First consider a rotatory-reflexion of angle α whose axis is a and whose plane is σ. Take any point P and through it draw a' parallel to a, and σ' parallel to σ. Let S, S' denote reflexions in σ and σ', and let $A(\alpha)$, $A'(\alpha)$ denote rotations through α about a and a'.

Then $S.S' =$ some translation T_1 (§ 9).

$\therefore \quad S = T_1.S'$, since $S'^2 = 1$.

Again $A(\alpha) = A'(\alpha).T_2$, where T_2 is some translation (p. 147). Therefore the rotatory-reflexion about a and σ, namely, $A(\alpha).S = A'(\alpha).T_2.T_1.S' = A'(\alpha).T'.S'$, where T' is some translation $= T.A'(\alpha).S'$, where T is some translation (see

§ 7, p. 151) = a similar rotatory-reflexion about a' and σ' multiplied by some translation or other.

Similarly $A(a).S = A'(a).S'.\mathsf{T}$ where T is some other translation.

Now consider the resultant of a screw $B_{\beta,t}$ and the rotatory-reflexion $A(a).S$.

Let $B'(\beta)$ be a rotation through β about a line through the point P parallel to the axis of $B_{\beta,t}$. Then $B_{\beta,t} = B'(\beta).T''$, where T'' is some translation (p. 151).

Therefore

$$B_{\beta,t}.A(a).S = B'(\beta).T''.T.A'(a).S' = T'.B'(\beta).A'(a).S^*$$

where T' is some translation or other.

This can be immediately extended, and we obtain:—

The resultant R of any number of screws and rotatory-reflexions is found by multiplying by some translation or other the resultant R' of rotations and rotatory-reflexions of the same angles about parallel axes and planes through any given point.

In this theorem rotations are included as particular cases of screws, and reflexions and inversions as particular cases of rotatory-reflexions.

We may put the theorem in the form $R = T.R' = R'.\mathsf{T}$.

We have shown in Part I how to obtain R'; the *practical* method of finding the resultant of any number of operations is exactly similar to that given on p. 151 for finding the resultant of any number of screws.

§ 11. There are two special cases of the above theorem whose simplicity and importance demand further attention. Independent proofs similar to that given for reflexions in two parallel planes (§ 9) can be at once obtained. They are:—

(1) The resultant of two inversions I, I_1 about points H, H_1 is a translation T represented by $2HH_1$.

Since $I^2 = I_1^2 = 1$ and $I.I_1 = T$; therefore $I = T.I_1$, $I_1 = I.T$.

(2) The resultant of a reflexion S in a plane σ and an inversion I_1 about a point H_1 is a screw E about the line HH_1 of angle π and translation $2HH_1$, where H is the foot of the perpendicular from H_1 on σ.

Since $S^2 = I_1^2 = 1$ and $S.I_1 = E$; therefore $I_1 = S.E$, $S = E.I_1$.

* See § 7 on p. 151.

§ 12. If A, B are any two operations, then $A^{-1}.B.A$ is called *the result of transforming B by A*.

Let L denote the axis* of the operation B, and let L' be the position into which L is brought by the operation A.

Now let *any* point c' be brought to c by A^{-1},
and let c be brought to d by B,
and let d be brought to d' by A.

Then c' is brought to d' by $A^{-1}.B.A$.

Now since the point c is brought to c' by A,
and the point d is brought to d' by A,
and the axis L* is brought to L' by A;

therefore the figure L, c, d is congruent or enantiomorphous to L', c', d' according as A is of the first or second sort.

Now B acting about L brings c to d; and therefore an operation acting about L' precisely similar to B (but whose angle of rotation is equal or equal and opposite to that of B, according as A is of the first or second sort) must bring c' to d'.

But $A^{-1}.B.A$ also brings c' to d', and the above is true for all positions of the point c'. Hence $A^{-1}.B.A$ must be identical with the operation similar to B (but whose angle of rotation is equal or equal and opposite to that of B, according as A is of the first or second sort) acting about the position into which the axis* of the operation B is brought by the operation A.

For example, if B is a right-handed screw, $A^{-1}.B.A$ is a screw of the same translation and angle, right-handed or left-handed, according as A is an operation of the first or second sort.

If an operation A brings a figure U to coincidence with U', U' is said to be the result of transforming U by A.

* Or the system axis + plane if B is of the second sort.

CHAPTER XVII

INFINITE GROUPS OF MOVEMENTS.

§ 1. *Every group of movements which brings a collection of molecules forming a crystal into self-coincidence has as a subgroup one of the fourteen translation-groups considered in chapter* XIV.

This follows from the definition of crystalline media given on p. 113. For if O, A be two points of the medium not very close together, then there must be a point A' such that AA' is small compared with OA, and such that the medium considered as a geometrical figure bears exactly the same relation to A' that it does to O; that is, such that the translation OA' brings the crystal to self-coincidence. Similarly, taking two points B, C not too close to O, and such that OA, OB, OC are not nearly coplanar, we have translations belonging to the group of movements which bring the medium to self-coincidence, represented by lines OB', OC' differing little from OB, OC in magnitude and position. We have then three translations OA', OB', OC' belonging to the group which are not coplanar, and the group can contain no infinitesimal translation; therefore the group must include a subgroup of translations of one of the fourteen kinds discussed on pp. 132 to 141.

§ 2. If a line l of given length and direction is brought to coincide with l' by any operation H, then the translation represented by l is said to be brought by H into coincidence with the translation represented by l'. If H brings every member of any system of translations into coincidence with itself or some other member of the system, H is said to bring the system of translations to self-coincidence.

A translation evidently brings any translation to coincidence with itself.

Any operation of a group must bring the system formed by all the translations of the group to self-coincidence: this is a particular case of the theorem that 'the result of trans-

forming any operation of a group by some other operation of the group itself belongs to the group *.'

§ 3. Two operations are said to be *isomorphous* if one can be derived from the other by multiplying by a translation.

If every operation of a group Γ is isomorphous with some operation of a group Γ' and vice versa, then Γ and Γ' are said to be *isomorphous*.

If L, M are two isomorphous operations, and T is a translation such that $L = M.T$; then $T = M^{-1}.L$, $M = L.T^{-1}$. Hence if two of the operations L, M, T belong to a group, so does the third; therefore:—

Any series of isomorphous operations of a group is obtained by combining any one of the operations with the translations of the group.

Suppose then that every operation of *any* group Γ is isomorphous with one of the operations $A', B', C', D', \ldots$ belonging to Γ.

We proved on pp. 151 and 154 that *every* operation is isomorphous with some operation which leaves a point P (chosen arbitrarily) unmoved, and that the result of combining any number of a series of operations such as $A', B', C', D', \ldots$ is isomorphous with the result of combining the corresponding members of the series $A, B, C, D, \ldots$; where $A, B, C, D, \ldots$ are operations leaving the point P unmoved and isomorphous with $A', B', C', D', \ldots$ respectively. But the result of combining any operations of the series $A', B', C', D', \ldots$ is an operation isomorphous with some member of the series; therefore the result of combining any operations of the series $A, B, C, D, \ldots$ is a member of this series, and hence $A, B, C, D, \ldots$ form a group. This group (G) consists solely of operations which leave a point unmoved; it is a *point-group* (see p. 48). We have then:—

Every group is isomorphous with a point-group.

It is evident from the above reasoning that those operations of Γ which are isomorphous with the operations of any subgroup of G form a subgroup of Γ, and vice versa.

§ 4. Let L' be *any* operation of Γ, and L the operation of G isomorphous with L'; then $L' = T.L$ where T is some translation. Now L' brings the system of translations (Γ_τ) of Γ into self-coincidence, and so does T; therefore L must do the same.

* For if A and B belong to a group, so does $A^{-1}.B.A$.

Hence G is a group of operations bringing the system Γ_r into self-coincidence.

Now suppose that Γ is a group of the kind considered in § 1 with three finite independent translations and no infinitesimal translations. Such a group will be called a *space-group*. Then the system Γ_r can be represented by a lattice ω, one of whose points is the point left unmoved by G. In this case every operation of G must bring ω to self-coincidence. Hence G can only contain rotations and rotatory-reflexions through 2π, π, $\frac{2\pi}{3}$, $\frac{\pi}{2}$, or $\frac{\pi}{3}$ (p. 132); that is, G is one of the groups discussed in chapters v and vi. Hence:—

Every space-group is isomorphous with one of the thirty-two point-groups consistent with the law of rational indices, and all the operations of the point-group are symmetry-operations of some lattice representing the system of translations of the space-group.

It follows that *the angle of any screw or rotatory-reflexion of a space-group is* 2π, π, $\frac{2\pi}{3}$, $\frac{\pi}{2}$, or $\frac{\pi}{3}$.

In future we shall mean by 'group' a 'space-group,' unless the contrary is expressly stated.

We see from the above that if G is a merohedral group contained in any system (triclinic, monoclinic, tetragonal, rhombohedral, hexagonal, or regular), then the symmetry of ω must be at least that of the holohedry of the same system.

Thus if G is C_{2v}. Γ_r must be represented by one of the lattices Γ_o, Γ_o', Γ_o'', Γ_o''' (p. 136). It would be consistent with the above reasoning if Γ_r were represented by Γ_t, Γ_t', Γ_r, Γ_r', or Γ_r''; but in this case the group Γ would only be a specialized form of a group whose translation-group is represented by Γ_o, Γ_o', Γ_o'', or Γ_o''', and would not be considered as distinct from it.

We may say in general then that 'the symmetry of ω is the holohedry of that system of which G is a merohedry.' An exception arises in the case of the rhombohedral system; for, though Γ_h has all the symmetry-elements of Γ_{rh}, yet Γ_h is not a specialized form of Γ_{rh}. For the translation-group represented by Γ_h is obtained by combining that represented by Γ_{rh} in the special case in which the translations OA_1, OA_2, OA_3 (Fig. 75) are parallel to a plane with a translation *not belonging to this specialized form of* Γ_{rh} perpendicular to that plane. Thus space-groups isomorphous with a point-group of the rhombohedral system whose translation-groups are

represented by Γ_h are essentially different from those isomorphous with the same point-groups whose translation-groups are represented by Γ_{rh}.

§ 5. Since any symmetry-axis of the lattice ω through one of its points passes through a row of its points, and any symmetry-plane (or the plane of any symmetry-operation of the second sort) through one of its points passes through a net of its points, and since the axes or planes of any two isomorphous operations are parallel; therefore any rotation-axis, screw-axis, or axis of rotatory-reflexion of the group Γ is parallel to a translation of Γ, and any symmetry-plane, gliding-plane, or plane of rotatory-reflexion is parallel to two independent translations of Γ.

Let $C_{\theta, t}$ be a screw of Γ, θ being a positive angle such that no screw of Γ about the same axis has an angle lying between o and θ. Let $n\theta = 2\pi$; then $n = 2, 3, 4$, or 6. Now, since $C_{\theta, t}$ is an operation of Γ, so is $(C_{\theta, t})^n = C_{n\theta, nt} = C_{o, nt}$ = a translation nt parallel to the axis of the screw. Let 2τ be the primitive translation of Γ parallel to this axis; then $nt = 2m\tau$ (where m is an integer), or $t = \frac{2m\tau}{n}$.

If t does not lie between o and $\frac{2(n-1)\tau}{n}$ (inclusive), we can find (by combining $C_{\theta, t}$ with some multiple of 2τ) a screw $C_{\theta, t'}$ contained in Γ such that t' lies between these limits, and is therefore one of the quantities $o, \frac{2\tau}{n}, \frac{4\tau}{n}, \frac{6\tau}{n}, \ldots, \frac{2(n-1)\tau}{n}$.

Such a movement $C_{\theta, t'}$ is called the *reduced* movement corresponding to the axis of $C_{\theta, t}$.

There can only be one reduced movement corresponding to a given axis, for if $C_{\theta, t'}$, $C_{\theta, t''}$ are two movements belonging to the group, then $C_{\theta, t'} \cdot (C_{\theta, t''})^{-1}$ belongs to the group and is equivalent to a translation $t' - t''$ parallel to the axis. But since 2τ is the *primitive* translation in this direction t' and t'' must be either equal or must differ by a multiple of 2τ, and therefore t' and t'' cannot both be included in the series $o, \frac{2\tau}{n}, \frac{4\tau}{n}, \frac{6\tau}{n}, \ldots, \frac{2(n-1)\tau}{n}$ unless they are equal.

The 'gliding-reflexion' $S(t)$, which consists of a reflexion in a plane σ combined with a translation t parallel to that plane, has similar properties.

Since the reflexion and translation of $S(t)$ are permutable

(p. 150, note), and the square of a simple reflexion = 1, therefore $\{S(t)\}^2$ is equivalent to a translation $2t$ parallel to the plane σ. If then $S(t)$ belongs to a group of movements Γ so does the translation $2t$.

Let $2\tau_1$, $2\tau_2$ be a primitive pair of translations of Γ parallel to σ. Then $2t = 2p_1\tau_1 + 2p_2\tau_2$ where p_1 and p_2 are integers, or $t = p_1\tau_1 + p_2\tau_2$.

Combining $S(t)$ with multiples of $2\tau_1$ and $2\tau_2$ we see that there is always some operation $S(t')$ belonging to Γ such that t' is one of the quantities 0, τ_1, τ_2, $\tau_1+\tau_2$. Such an operation is called the *reduced* operation corresponding to the pair $2\tau_1$, $2\tau_2$. As in the case of a screw, it may be readily shown that there is only *one* reduced operation corresponding to each primitive pair in the plane σ.

§ 6. Let A_{θ, t_a} be a screw of a group Γ about an axis a, and let T be any translation of Γ whose components parallel and perpendicular to a are τ_a, τ_n. Then $A_{\theta, t_a} \cdot T$ = a screw $B_{\theta, t_a + \tau_a}$ about some axis b parallel to a. Thus the 'reduced' movements corresponding to the axes a and b have the same angle, and have or have not the same translation according as τ_a is or is not a multiple of the primitive translation of Γ parallel to a. Hence:—

All reduced screws of a space-group which are isomorphous with a single rotation of the corresponding point-group have or have not the same translation according as the space-group does not or does contain a translation whose component parallel to the axes of the screws is less than the primitive translation of the space-group in this direction.

Consider again a set of screws contained in Γ and isomorphous with a single operation of the corresponding point-group. We may take as a primitive triplet of the translations 2τ, $2\tau_1$, $2\tau_2$ where 2τ is the primitive translation parallel to the axes of the screws. Let e be the axis of any one of the screws, and let e_1, e_2, e_3 be the positions into which e is brought by the translations $2\tau_1$, $2\tau_2$, $2\tau_1 + 2\tau_2$. Let e, e_1, e_2, e_3 be cut in the points E, E_1, E_2, E_3 by a plane perpendicular to them all. Let the axis of any one of the isomorphous set of screws be f; then it is evident that there must be a similar screw round some axis which meets the surface or circumference of the parallelogram $E\,E_1\,E_2\,E_3$ and which is obtained from f by transforming it by some translation of the form $2p_1\tau_1 + 2p_2\tau_2$ (where p_1 and p_2 are integers); hence:—

To find all the screws isomorphous with a single rotation

of the corresponding point-group it is only necessary to find all whose axes meet a certain parallelogram, and to transform them by the translations of the group.

§ 7. We shall now prove certain theorems by means of which one group can be derived from another. Suppose the operations of a point-group G_1 are

$$1, L_1, L_2, L_3, \ldots .$$

Let H be an operation bringing the system of symmetry-elements of G_1 to self-coincidence; then $H^2, H^3, \ldots$ also bring this system to self-coincidence.

Since the system can be thus brought to self-coincidence in only a finite number of ways some power of H is unity *, that is, some power of H is an operation of G. Let H^m $(= L_k)$ be the lowest power of H which is an operation of G.

Then
$$\left.\begin{array}{llll} 1, & L_1, & L_2, & L_3, \ldots \\ H, & H.L_1, & H.L_2, & H.L_3, \ldots \\ H^2, & H^2.L_1, & H^2.L_2, & H^2.L_3, \ldots \\ \ldots & \ldots & \ldots & \ldots \\ H^{m-1}, & H^{m-1}.L_1, & H^{m-1}.L_2, & H^{m-1}.L_3, \ldots \end{array}\right\} \quad \text{(i)}$$

form a group G.

For the product of any two of these operations $H^\alpha . L_p$ and $H^\beta . L_q$ (say) $= H^\alpha . L_p . H^\beta . L_q = H^{\alpha+\beta} . H^{-\beta} . L_p . H^\beta . L_q$.

Now, since H^β brings the system of symmetry-elements of G_1 into self-coincidence, the result of transforming any operation of G_1 by H^β is another operation of G_1 (p. 156).

Therefore the product of $H^\alpha . L_p$ and $H^\beta . L_q = H^{\alpha+\beta} L_i . L_q = H^{\alpha+\beta} . L_u = H^{\alpha+\beta-m} L_h . L_u = H^{\alpha+\beta-m} . L_k$. Hence the product is a member of the series (i); for either $\alpha+\beta$ or $\alpha+\beta-m$ is an integer lying between 0 and $(m-1)$ inclusive.

It is readily seen that no two members of the series (i) are identical. For if $H^\alpha . L_p = H^\beta . L_q$, $H^{\alpha-\beta} = L_q . L_p^{-1} = L_w$, which is only possible if $\alpha = \beta$ and $L_w = 1$.

We may show that the operations obtained by multiplying $1, L_1, L_2, L_3, \ldots$ by $1, H, H^2, \ldots, H^{m-1}$ on the *right*-hand side form the same group. For $L_v . H^\gamma = H^\gamma . H^{-\gamma} . L_v . H^\gamma = H^\gamma . L_s$.

The case of greatest importance is that for which $m = 2$; the theorem then becomes:—

'If $1, L_1, L_2, L_3, \ldots$ are the operations of a group G_1, and H is an operation which brings the system of symmetry-elements

* The angle of the operation H is assumed finite.

of G_1 to self-coincidence and is such that H^2 is an operation of G_1, then

$$\left.\begin{matrix} 1, & L_1, & L_2, & L_3, & \dots \\ H, & H.L_1, & H.L_2, & H.L_3, & \dots \end{matrix}\right\} \text{ form a group.'}$$

In chapter vi we used the equally sound but longer process of verifying this fact in each particular case.

§ 8. If V is any space- or point-group and K any operation, we shall in future express by the symbols $[K, V]$, $[V, K]$ the series of operations formed by multiplying all the operations of V by K on the left- or right-hand side respectively.

Now let Γ_1 be a space-group isomorphous with G_1 (of § 7), and let L_e' be any operation of Γ_1 isomorphous with L_e; then the series of operations of Γ_1 which are isomorphous with L_e is $[L_e', \Gamma_\tau]$, where Γ_τ is the group of translations of Γ_1 (p. 158).

Let now H' be an operation isomorphous with H which brings the system of symmetry-elements of Γ_1 to self-coincidence.

Then H'^m is isomorphous with some operation of G_1; take H' such that H'^m is an operation of Γ_1 ($= L_k'$ say).

We can prove now that the series of operations

$$\left.\begin{matrix} \Gamma_\tau, & [L_1', \Gamma_\tau], & [L_2', \Gamma_\tau], & [L_3', \Gamma_\tau], & \dots \\ [H', \Gamma_\tau], & [H'.L_1', \Gamma_\tau], & [H'.L_2', \Gamma_\tau], & [H'.L_3', \Gamma_\tau], & \dots \\ \dots & \dots & \dots & \dots & \dots \\ [H'^{m-1}, \Gamma_\tau], & [H'^{m-1}.L_1', \Gamma_\tau], & [H'^{m-1}.L_2', \Gamma_\tau], & [H'^{m-1}.L_3', \Gamma_\tau], & \dots \end{matrix}\right\} \text{(ii)},$$

each of which is isomorphous with some operation of (i), forms a group Γ isomorphous with G.

For the product of any two of the operations (ii), e.g. $H'^{\alpha}.L_p'.T_1$ and $H'^{\beta}.L_q'.T_2 = H'^{\alpha+\beta}.H'^{-\beta}.L_{p_1}'.H'^{\beta}.L_q'.T_2$ (where $L_{p_1}' = L_p'.T_1$) $= H'^{\alpha+\beta}.L_n'.T_3 = H'^{\alpha+\beta-m}.L_k'.L_n'.T_3$ $= H'^{\alpha+\beta-m}.L_k'.T_4$ (where L_n', L_k', &c., are members of the first row of (ii) and T_1, T_2, T_3, T_4 some translations contained in Γ_τ) = some member of the series (ii), since either $\alpha+\beta$ or $\alpha+\beta-m$ is an integer lying between 0 and $(m-1)$ inclusive. It is readily seen that no two members of the series (ii) are identical.

Conversely *every* group Γ isomorphous with G can be obtained by multiplying the operations of the subgroup Γ_1 consisting of those operations of Γ which are isomorphous with the operations of G_1 by some operation H' which is isomorphous with H, brings the system of symmetry-elements of Γ_1 into self-coincidence, and is such that the lowest power of H'

which is isomorphous with an operation of Γ_1 is an operation of Γ_1.

For let H' be *any* operation of Γ isomorphous with H. Then, firstly, H' is a symmetry-operation of the axes and planes of Γ_1. For $H'^{-1}.L'_f.H'$ (where L'_f is any operation of Γ_1) is an operation of Γ and is isomorphous with $H^{-1}.L_f.H$, that is, with some operation of G_1; therefore $H'^{-1}.L'_f.H'$ is an operation of Γ_1. Hence H' is a symmetry-operation of the system of elements of Γ_1.

Secondly, the lowest power of H' which is isomorphous with an operation of Γ_1 is an operation of Γ and is therefore an operation of Γ_1.

Thirdly, the group formed by combining the operations of Γ_1 with H' is identical with Γ. For since by combining the operations of G_1 with H we get every operation of G, therefore by combining some operation of Γ_1 with H' we get an operation of Γ isomorphous with any given operation of Γ. But all isomorphous operations of Γ are obtained by multiplying any one of them by the operations of the translation-group (Γ_τ) of Γ. Now evidently any space-group which is a subgroup of Γ must have the same translation-group as Γ; so that the operations of Γ_τ are included in those of Γ_1. We can therefore get *every* operation of Γ by combining H' with operations of Γ_1. Hence we have the following theorem:—

A point-group G is formed by combining the operations of a subgroup G_1 with an operation H which brings the system of symmetry-elements of G_1 to self-coincidence. Γ_1 is any space-group isomorphous with G_1; and H' is any operation isomorphous with H which brings the system of symmetry-elements of Γ_1 to self-coincidence, and is such that the lowest power of H' which is isomorphous with an operation of Γ_1 is itself an operation of Γ_1. Then a group isomorphous with G is obtained by combining the operations of Γ_1 with H'†; and conversely all space-groups isomorphous with G can be obtained in a similar way.*

This theorem is of fundamental importance.

* This last condition is always fulfilled if H' is an inversion or a simple reflexion. When H' is a gliding-reflexion $S(t)$ in a plane which is not parallel to any symmetry plane of G_1 the condition is fulfilled if and only if $2t$ is a translation of Γ_1. When H' is a screw $A_{\frac{2\pi}{n},t}$ about an axis which is not parallel to any axis of G_1 the condition is fulfilled if and only if nt is a translation of Γ_1.

† The groups obtained by taking various operations such as H' are not necessarily all distinct. Groups whose subgroups Γ_1 are distinct are evidently themselves distinct.

The operations of the series (ii) are all different; the group formed by them will be denoted by the symbol $\{H', \Gamma_1\}$.

§ 9. Let H' be an operation of the second sort which brings the system of symmetry-elements of a group Γ_1 into self-coincidence. Suppose an axis a to be brought into coincidence with an axis b by H'.

Now the result of transforming a right-handed screw by an operation of the second sort is a similar left-handed screw * (p. 156). Hence if b is identical with a, the group must contain similar right-handed and left-handed screws about a. This is only possible if the translation of these screws is either zero or half a translation of the group parallel to a. If however b is not identical with a, Γ_1 is such that to every right-handed screw about any axis there corresponds a similar left-handed screw about some other axis. Hence:—

If the system of elements of a group Γ_1 is brought to self-coincidence by an operation of the second sort, either the axes of Γ_1 are rotation-axes and screw-axes whose translation is half a translation of Γ_1 in the direction of the axes, or else similar right-handed and left-handed screws of Γ_1 occur in pairs.

This theorem is also of great importance: it shows that no space-group Γ can be obtained by combining an operation of the second sort with the operations of any group Γ_1 whose symmetry-elements do not satisfy the above condition.

§ 10. Let H_1', H_2' be two symmetry-operations of the system of elements of any group Γ_1; let $H_1'^m$, $H_2'^n$ be the lowest powers of H_1' and H_2' respectively, which are isomorphous with an operation of Γ_1, and let $H_1'^m$, $H_2'^n$ be operations of Γ_1.

Then if $\{H_1', \Gamma_1\}$ and $\{H_2', \Gamma_1\}$ are identical † $(= \Gamma)$, Γ contains both H_1' and H_2'; moreover, since in this case $\{H_1', \Gamma_1\}$ and $\{H_2', \Gamma_1\}$ must contain the same number of distinct sets of isomorphous operations, we must have $m = n$ (cf. p. 163).

Again if $\{H_1', \Gamma_1\}$ contains H_2' and *if m is prime*, $\{H_1', \Gamma_1\}$ and $\{H_2', \Gamma_1\}$ are identical and $m = n$.

For if $\{H_1', \Gamma_1\}$ contains H_2', $H_1'^r . L' = H_2'$ where L' is some operation of Γ_1 and r some positive integer $< m$ (p. 163).

* That is, a screw of the same translation and of equal and opposite angle of rotation.

† This identity of the two groups is expressed by the symbol $=$. Thus $\{H_1', \Gamma_1\} = \{H_2', \Gamma_1\}$.

Therefore $\{H_2', \Gamma_1\} = \{H_1'^r . L', \Gamma_1\} = \{H_1'^r, \Gamma_1\}$.

Now since r is prime to m the series

$$\Gamma_1, [H_1'^r, \Gamma_1], [H_1'^{2r}, \Gamma_1], \ldots, [H_1'^{(m-1)r}, \Gamma_1]$$

is evidently identical with

$$\Gamma_1, [H_1', \Gamma_1], [H_1'^2, \Gamma_1], \ldots, [H_1'^{(m-1)}, \Gamma_1]^*;$$

for $0, r, 2r, \ldots, (m-1)r$ leave different remainders when divided by m.

Hence $\{H_1', \Gamma_1\} = \{H_1'^r, \Gamma_1\} = \{H_2', \Gamma_1\}$, and therefore $m = n$.

Similarly if $\{H_2', \Gamma_1\}$ contains H_1' and n is prime, $\{H_2', \Gamma_1\}$ and $\{H_1', \Gamma_1\}$ are identical and $n = m$.

The case of $m = n = 2$ is of special importance.

§ 11. Let a space-group isomorphous with a point-group G have two subgroups Γ_1, Γ_2 isomorphous with the point-groups G_1, G_2 respectively. Suppose that *every* operation of G is of the form $M . N$, where M is an operation of G_1 and N an operation of G_2. Then evidently *every* operation of Γ is of the form $M' . N'$, where M' is an operation of Γ_1, and N' an operation of Γ_2.

Let H' be a symmetry-operation of the elements of Γ_1 and the elements of Γ_2. Then

$$H'^{-1} . M' . H' = M_1', \; H'^{-1} . N' . H' = N_1',$$

where M_1' is an operation of Γ_1 and N_1' is an operation of Γ_2.

Therefore the result of transforming any operation $M' . N'$ of Γ by H' is $H'^{-1} . M' . N' . H' = H'^{-1} . M' . H' . H'^{-1} . N' . H' = M_1' . N_1'$ which is another operation of Γ.

Therefore H' is a symmetry-operation of the elements of Γ.

§ 12. If we subject *any* point P to all the movements of a space-group Γ isomorphous with a point-group G, we obtain an infinite series of 'equivalent' points. There is no difficulty in obtaining the coordinates of all the points of such a series referred to any axes whose origin is the centre of G. Take one operation of Γ isomorphous with each operation $L_1, L_2, L_3, \ldots$ of G; suppose these to be $L_1', L_2', L_3', \ldots$. Let L_{i_n}' be *any* operation of Γ isomorphous with L_i'; then $L_{i_n}' = L_i' . T$ where T is some translation of Γ, and $L_i' = L_i . T_1$ where T_1 is some translation which may readily be obtained in each particular case. Then any point P is brought by L_i to a point whose coordinates are known from chapter vii; adding on to these

* The operations are not in general arranged in the same order in the two series.

coordinates the components of T_1 along the axes we obtain the coordinates of the point to which P is brought by L_1'. Similarly we obtain the coordinates of the points to which P is brought by each of the operations L_1', L_2', L_3', ... To obtain the coordinates of all the points to which P is brought by the operations of Γ we have only to add to these the components of all the translations of Γ.

For the application of this method to each one of the 230 possible space-groups we must refer the reader to Schoenflies' "Krystallsysteme und Krystallstructur."

§ 13. We are at last in a position to find all possible space-groups which have a crystallographic application. All those which are isomorphous with a given point-group form a *class* which is named after the point-group. A space-group is said to belong to the same *system* as the isomorphous point-group.

We shall denote any space-group by the same symbol as the isomorphous point-group with the addition of a positive integral index.

Thus, for example, C_{4v}^m denotes '*any** space-group isomorphous with the point-group C_{4v},' or '*any** space-group of the hemimorphic hemihedry of the tetragonal system'; while C_{4v}^1, C_{4v}^2, C_{4v}^3, ... denote *definite* space-groups isomorphous with C_{4v}.

A group is completely given when the arrangement of all its symmetry-elements (i. e. its screw-axes, gliding-planes, &c.) relatively to its translations is given.

* Or '*some* space-group,' '*a* space-group,' &c.

CHAPTER XVIII

TRICLINIC AND MONOCLINIC GROUPS.

§ 1. Triclinic Hemihedry.

A group of this class can only have operations isomorphous with unity, since this is the only operation of the isomorphous point-group C_1; that is, it can only have translations. Considering for this purpose the fourteen translation-groups of chapter xiv as merely specialized forms of the most general of them (Γ_{tr})*, we may say that *there is only one group of the triclinic hemihedral class.* We denote it by C_1^1 or Γ_{tr}.

§ 2. Triclinic Holohedry.

Since the point-group C_i is obtained by multiplying the operation of C_1 by an inversion, therefore all groups C_i^m isomorphous with C_i are obtained by multiplying the operations of Γ_{tr} by an operation isomorphous with an inversion, that is, by an inversion †. Now *every* group of translations is brought to self-coincidence by an inversion about *any* point P; there is therefore one and only one distinct group of this class, namely

$$C_i^1 = \{\Gamma_{tr}, I\}.$$

We may represent the translations of C_i^1 by a lattice, one of whose points is at P; let Q be any other point of the lattice; then an inversion about P followed by a translation represented by PQ is equivalent to an inversion about a point bisecting the line PQ (p. 155). Hence the symmetry-elements of C_i^1 are a series of centres of symmetry at the points of a lattice representing Γ_{tr} and at the centres of all straight lines joining two such points ‡.

* We shall in future denote a translation-group by the same symbol as the lattice representing it.

† We pointed out on p. 155 that every operation isomorphous with an inversion is an inversion.

‡ That is, at the vertices, centres, centres of the faces, and centres of the edges of the parallelepipeda of Fig. 66.

§ 3. MONOCLINIC HEMIHEDRY.

A group of the monoclinic system has either Γ_m or Γ_m' as its translation-group (p. 133).

Since the point-group C_s is obtained by multiplying the operations of C_1 by a reflexion, therefore groups C_s^m isomorphous with C_s are obtained by combining the operations of C_1^1 with a gliding-reflexion $S(t)$ in some plane σ (where t is zero or half a translation of C_1^1 in the plane σ), which brings the system of translations C_1^1 to self-coincidence. This last condition is only satisfied if C_1^1 has the specialized form Γ_m or Γ_m' and if σ is perpendicular to the translation $2\tau_z$ of Γ_m or Γ_m' (using the notation of p. 133).

Suppose now C_1^1 identical with Γ_m.

First, take $t = 0$. We have then a group,

$$C_s^1 = \{\Gamma_m, S\}.$$

Since S followed by the translation $2p\tau_z$ is equivalent to a reflexion in a plane parallel to σ and at a distance $p\tau_z$ from it, C_s^1 has a 'set' of symmetry-planes of interval τ_z (p. 115).

Secondly, take $t \neq 0$. We have then a group

$$C_s^2 = \{\Gamma_m, S(t)\},$$

$2t$ is a translation parallel to σ; two groups $\{\Gamma_m, S(t_1)\}$, $\{\Gamma_m, S(t_2)\}$ are not geometrically distinguishable unless t_1 or t_2 (not both) $= 0$.

$S(t)$ followed by the translation $2p\tau_z$ is evidently equivalent to an operation exactly similar to $S(t)$ in a plane whose distance from σ is $p\tau_z$; hence C_s^2 has a set of similar gliding-planes of interval τ_z.

Since Γ_m has no translation whose component perpendicular to σ is not a multiple of $2\tau_z$, therefore C_s^1 and C_s^2 have no symmetry-elements other than those mentioned.

§ 4. Next consider the result of combining the translation-group Γ_m' with $S(t)$.

A primitive triplet of Γ_m' may be taken as $2\tau_e$, $2\tau_f$, $\tau_e+\tau_f+\tau_z$, where $2\tau_e$, $2\tau_f$ are a primitive pair in the plane perpendicular to the symmetry-axis of Γ_m' (and therefore parallel to σ) and $2\tau_z$ is the primitive translation perpendicular to the plane*.

We need only consider the *reduced* operation $S(t)$; it is of the form $S(a_1\tau_e+a_2\tau_f)$, where a_1, a_2 have any one of the sets of values 0, 0; 0, 1; 1, 0; 1, 1.

Now any operation of Γ_m' is of the form $2p_1\tau_e+2p_2\tau_f+p_3(\tau_e+\tau_f+\tau_z)$. If p_3 is even the result of combining this with

* We still use the notation of p. 133.

$S(a_1\tau_e + a_2\tau_f)$ and the translation $-(2p_1+p_3)\tau_e - (2p_2+p_3)\tau_f$ is an operation $S(a_1\tau_e + a_2\tau_f)$ in a plane whose distance from σ is $\frac{p_3}{2}\tau_z$.

The group $\{\Gamma'_m, S(a_1\tau_e + a_2\tau_f)\}$ therefore contains a series of operations similar to $S(a_1\tau_e + a_2\tau_f)$ in a 'set' of planes of interval τ_z.

If p_3 is odd the result of combining

$2p_1\tau_e + 2p_2\tau_f + p_3(\tau_e + \tau_f + \tau_z)$ with $S(a_1\tau_e + a_2\tau_f)$

and the translation $-(2p_1+p_3-1)\tau_e - (2p_2+p_3-1)\tau_f$ is an operation $S(\overline{a_1+1}\,\tau_e + \overline{a_2+1}\,\tau_f)$ in a plane whose distance from σ is $\frac{p_3}{2}\tau_z$.

The group $\{\Gamma'_m, S(a_1\tau_e + a_2\tau_f)\}$ therefore contains a series of operations of the type $S(\overline{a_1+1}\,\tau_e + \overline{a_2+1}\,\tau_f)$ in a set of planes of interval τ_z, which lie halfway between the gliding-planes whose operation is $S(a_1\tau_e + a_2\tau_f)$.

Hence, if $a_1 = a_2 = 0$, the group $\{\Gamma'_m, S(a_1\tau_e + a_2\tau_f)\}$ has a series of operations S in a set of planes of interval τ_z, and a series of operations $S(\tau_e + \tau_f)$ in the planes halfway between them.

If $a_1 = a_2 = 1$, the group has a series of operations $S(\tau_e + \tau_f)$ in a set of planes of interval τ_z, and a series of operations $S(2\tau_e + 2\tau_f)$, and therefore a series of operations S^* in the planes halfway between them.

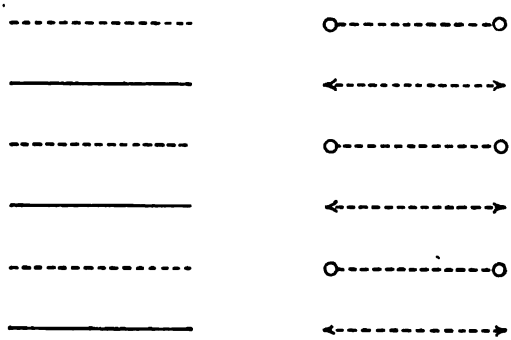

Fig. 84. C_s^3.

Fig. 85. C_s^4.

These two cases are not geometrically distinguishable, we denote them both by

$$C_s^3 = \{\Gamma'_m, S\}.$$

* For $2\tau_e + 2\tau_f$ is a translation of the group.

If $a_1 = 0$, $a_2 = 1$, the group has a series of operations $S(\tau_f)$ in a set of planes of interval τ_g, and a series of operations $S(\tau_e + 2\tau_f)$, and therefore a series of operations $S(\tau_e)$* in the planes halfway between them.

The case of $a_1 = 1$, $a_2 = 0$ is of course not geometrically distinguishable from this; we denote the group in this case by

$$C_s^4 = \{\Gamma_m', S(\tau)\}.$$

We may represent the groups C_s^3, C_s^4 diagrammatically in Figs. 84 and 85, if we take the symmetry-axis of Γ_m' vertical. If the plane of an operation $S(t)$ is perpendicular to the plane of the paper, its intersection with the paper will be denoted † by a simple line if $t = 0$; by a broken line if the direction of $t (\neq 0)$ is perpendicular to the plane of the paper; by a broken line with arrows if the direction of t is in the plane of the paper, and by a broken line with small circles if t has any other direction.

In Fig. 84 we have taken the direction of $\tau_e + \tau_f$ perpendicular to the plane of the paper. C_s^1 would be represented by the simple lines of Fig. 84 taken alone, C_s^2 by the broken lines. In Fig. 85 we have taken the direction of τ_e parallel to the plane of the paper. C_s^2 would be represented either by the broken lines alone, or by the broken lines with circles alone.

The distance between the extreme lines of either figure is $\frac{5\tau_g}{2}$.

§ 5. Monoclinic Hemimorphy.

Since the point-group C_2 is obtained by combining the operation of C_1 with a rotation through π about some axis, therefore the groups C_2^m are obtained by combining a translation-group with some screw $A_{\pi,t}$ which brings the system of translations of self-coincidence. To fulfil this last condition the translation-group must be either Γ_m or Γ_m' (cf. p. 159), and the axis of the screw (e) must be parallel to τ_g.

We may take the screw as a reduced operation; in this case $t = 0$ or τ_g (p. 160).

First consider the result of combining $A_{\pi,t}$ with Γ_m of which $2\tau_e$, $2\tau_f$, $2\tau_g$ are a primitive triplet.

Since there is no translation whose component in the direction of e is not a multiple of $2\tau_g$, the reduced screws about all axes parallel to e have the same translation (p. 161); that is, the group $\{\Gamma_m, A_{\pi,t}\}$ contains only rotation-

* For $2\tau_f$ is a translation of the group.

† This is Fedorow's method.

axes or only screw-axes, according as $t=0$ or τ_z. To find the distribution of these axes, suppose that e is perpendicular to the plane of the paper (Fig. 86) and meets it in E, and let EE_1, EE_2, EE_3 represent $2\tau_e$, $2\tau_f$, $2\tau_e+2\tau_f$ respectively. Then, since the translations bring the system of symmetry-axes to self-coincidence, axes similar to e pass through E_1, E_2, E_3. To find the whole system of axes it is only necessary to find those meeting the surface or circumference of the parallelogram $EE_1E_2E_3$, and to transform them by the translations of the group (p. 161). Suppose such an axis meets the parallelogram in P, then the screw about e, followed by the screw about a parallel axis through P, is equivalent to a translation whose component parallel to the plane of the

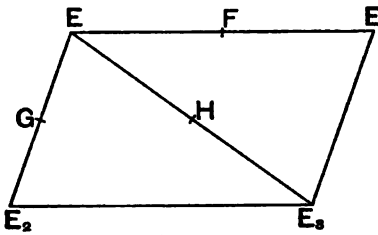

Fig. 86.

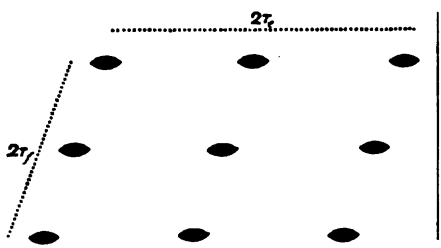

Fig. 87. C_2^1.

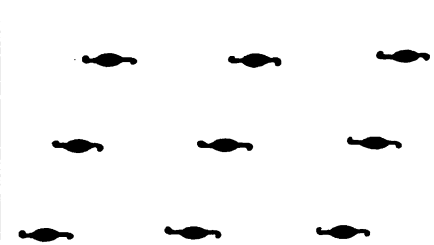

Fig. 88. C_2^2.

parallelogram is $2EP$. Therefore P must bisect a side or diagonal of the parallelogram.

We have then two groups,

$$C_2^1 = \{\Gamma_m, A_{\pi,0}\}, C_2^2 = \{\Gamma_m, A_{\pi,\tau_z}\},$$

the arrangement of whose axes is shown in Figs. 87 and 88 *, if we represent a 2-al rotation-axis perpendicular to the plane of the paper by ⬬ and a similar screw-axis by ⬭.

§ 6. Next consider the result of combining $A_{\pi,t}$ with Γ'_m, of which $2\tau_z$, $\tau_e+\tau_z$, $\tau_f+\tau_z$ are a primitive triplet.

* In these and all other similar figures we shall suppose ourselves looking from the positive towards the negative end of the axes; that is, the translation τ_z brings an object *closer* to us.

As before, let e be perpendicular to the plane of the paper and meet it in E; let EE_1, EE_2, EE_3 represent τ_e, τ_f, $\tau_e+\tau_f$ respectively.

Then, since the result of transforming $A_{\pi,t}$ by $\tau_e+\tau_z$, $\tau_f+\tau_z$, or $(\tau_e+\tau_z)+(\tau_f+\tau_z)$ is a similar operation of the group, axes exactly similar to e must pass through E_1, E_2, E_3. As before, we find all the axes by transforming those which meet the surface and circumference of $EE_1E_2E_3$ by the operations of Γ'_m; and, as before, we see that all axes meeting this parallelogram must bisect a side or diagonal. Now $A_{\pi,t}$ followed by the translation $\tau_e+\tau_z$ is equivalent to a screw of angle π and translation $t+\tau_z$ about the axis f parallel to e which bisects EE_1; and $A_{\pi,t}$ followed by the translation $(\tau_e+\tau_z)+(\tau_f+\tau_z)$ and the translation $-2\tau_z$ is equivalent to a screw of angle π and translation t about the axis h parallel to e which bisects EE_3. Hence, if $t=0$, we have rotation-axes through the vertices and centre of $EE_1E_2E_3$, and screw-axes of translation τ_z through the middle points of the sides.

The general arrangements of the axes of the group

$$C_2^3 = \{\Gamma'_m, A_{\pi,0}\}$$

is shown in Fig. 89, from which also it is evident that the group $\{\Gamma'_m, A_{\pi,\tau_z}\}$, which has rotation-axes through F and G, and screw-axes through E and H, is geometrically identical with C_2^3.

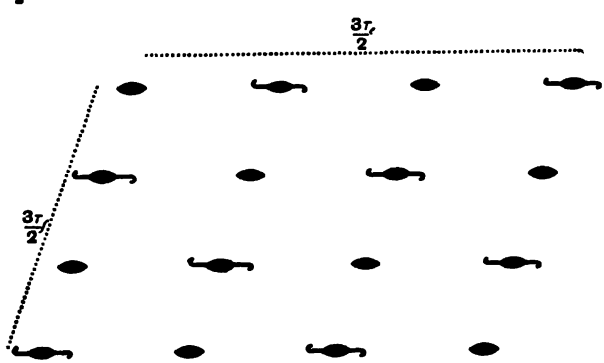

Fig. 89. C_2^3.

If we had taken $E_1E=\tau_f$, $EE_3=\tau_e$, $EE_2=\tau_e+\tau_f$ (instead of $EE_1=\tau_e$, $EE_2=\tau_f$, $EE_3=\tau_e+\tau_f$), we should again have an arrangement of axes similar to that of Fig. 86; but

now the axes of $\{\Gamma_m', A_{\pi,0}\}$ through E and G would be rotation-axes, and those through F and H screw-axes, that is, the rotation-axes would have passed through *adjacent* (instead of *opposite*) vertices of the parallelogram $EFGH$.

§ 7. Monoclinic Holohedry.

Since the point-group C_{2h} is obtained by multiplying the operations of C_2 by an inversion, therefore a group C_{2h}^m is obtained by multiplying any group C_2^m, which fulfils the conditions of p. 165, by an operation isomorphous with an inversion (i.e. by an inversion) which brings the system of symmetry-elements of C_2^m into self-coincidence; and all groups C_{2h}^m can be so obtained (p. 164).

All three groups, C_2^1, C_2^2, C_2^3, satisfy the condition of p. 165; their systems of symmetry-elements are evidently brought to self-coincidence by inversion about a point P if, and only if, P lies either in an axis, or halfway between two axes of the same sort (i.e. both rotation- or both screw-axes).

First, suppose P lies in an axis; we have then three groups

$$C_{2h}^1 = \{C_2^1, I\} = \{C_2^1, S\}. \quad \text{Subgroup } C_s^1.$$

$$C_{2h}^2 = \{C_2^2, I\} = \{C_2^2, S\}. \quad \text{Subgroup } C_s^1.$$

$$C_{2h}^3 = \{C_2^3, I\} = \{C_2^3, S\}. \quad \text{Subgroup } C_s^3.$$

Since C_{2h} has a subgroup C_s, every group C_{2h}^m has a subgroup C_s^m.

Now the operation $A_{\pi,t}$ about an axis e, followed by an inversion about a point in e is equivalent to a reflexion in a plane perpendicular to e. Therefore C_s^1 is the subgroup of C_{2h}^1, C_{2h}^2, and C_s^3 that of C_{2h}^3. We could have derived C_{2h}^1, C_{2h}^2, C_{2h}^3 from C_2^1, C_2^2, C_2^3 respectively, by combining their operations with a reflexion instead of with an inversion, if we had so chosen *.

Next, suppose P lies halfway between two axes of the same sort; we have then these three groups

$$C_{2h}^4 = \{C_2^1, I_1\} = \{C_2^1, S(t)\}. \quad \text{Subgroup } C_s^2.$$

$$C_{2h}^5 = \{C_2^2, I_1\} = \{C_2^2, S(t)\}. \quad \text{Subgroup } C_s^3.$$

$$C_{2h}^6 = \{C_2^3, I_1\} = \{C_3^2, S(t)\}. \quad \text{Subgroup } C_s^4.$$

These three groups contain no simple reflexions (for they

* Since $\{C_2^1, I\}$ contains the reflexion S, therefore $\{C_2^1, I\}$ and $\{C_2^1, S\}$ are identical; see bottom of p. 165.

are evidently distinct from C_{2h}^1, C_{2h}^2, C_{2h}^3, respectively), and must therefore contain gliding-reflexions, for they each have a subgroup C_s^m.

Evidently C_s^2 is the subgroup of C_{2h}^4 and C_{2h}^5, and C_s^4 that of C_{2h}^6.

The groups C_{2h}^m contain an infinite number of centres of symmetry. Their position is at once found by drawing a line PQ from P representing *any* translation of the group. Then the group has a centre of symmetry at the middle point of PQ (p. 155). It may be noted that in the case of C_{2h}^1, C_{2h}^2, C_{2h}^3 the centres of symmetry form a 'row' of points of interval τ_z on *every* axis.

CHAPTER XIX

ORTHORHOMBIC GROUPS.

§ 1. Orthorhombic Hemimorphy.

All orthorhombic groups are isomorphous with some orthorhombic point-group; their translation-groups are Γ_o, Γ_o', Γ_o'', or Γ_o'''. Hemimorphic groups are isomorphous with C_{2v}; and since C_{2v} may be derived from C_2 by multiplying by a reflexion in a plane through the axis of C_2, all groups C_{2v}^m may be derived by multiplying some group C_2^m by an operation $S(t)$* which brings the system of axes of C_2^m into self-coincidence. Such an operation can only be found if the translation-group of C_2^m has a specialized form, in fact, if Γ_m is specialized to Γ_o or Γ_o', or Γ_m' to Γ_o', Γ_o'', or Γ_o'''. Fig. 86 is consequently specialized in one of two ways; either the angle GEF is a right angle (Fig. 90), or else $EF = EG$ (Fig. 91).

We may denote as before the axes of C_2^m which pass through E, F, G, H by e, f, g, h respectively. We shall represent by the symbol **e** the series of axes obtained by transforming e by the translations; and **f**, **g**, **h** will have similar meanings.

If Fig. 86 is so specialized that GEF is a right angle (but $EF \neq EG$ in general), the plane of $S(t)$ must evidently be parallel to one of the sides of the parallelogram $EFGH$, and pass through an axis or lie halfway between two axes.

The plane through the axes e and f we shall denote by σ_0, that through g and h by σ_0', that through e and g by σ_1, that through f and h by σ_1', that through the centre of the parallelogram parallel to the side EF by σ_m, that through the centre parallel to the side EG by σ_{m_1} (Fig. 90).

If Fig. 86 is so specialized that $EF = EG$ (but GEF is not in general a right angle), the plane of $S(t)$ must evidently

* Including the case where $t = o$. The plane of $S(t)$ must be parallel to the axes of C_2^m.

pass through one of the axes and be parallel to a diagonal of the rhombus *EFGH*.

We shall denote the plane through e and h by σ_d, that through f and g by σ_{d_1}', that through e parallel to FG by σ_{d_1}, that through F parallel to EH by σ_d' (Fig. 91).

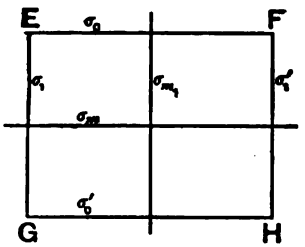

Fig. 90.

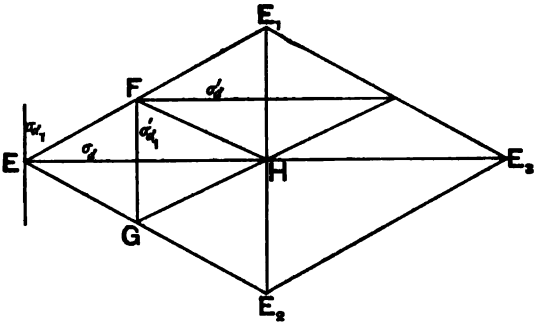

Fig. 91.

C_{2v} has two perpendicular symmetry-planes; therefore all groups C_{2v}^m have two series of gliding-planes at right angles to one another.

If the axes are arranged as in Fig. 90 these gliding-planes are parallel to σ_0 and σ_1 respectively; it is evident that we obtain all groups C_{2v}^m whose axes are so arranged if we always take the plane of $S(t)$ parallel to *one* of these, say σ_0.

If the axes are arranged as in Fig. 91 these gliding-planes are parallel to σ_d and σ_{d_1}; it is evident that we obtain all groups C_{2v}^m whose axes are so arranged if we always take the plane of $S(t)$ parallel to *one* of these, say σ_d.

§ 2. Consider first the groups C_{2v}^m whose translation-group is Γ_0. Since this is a specialized form of Γ_m, these groups must be derived from groups C_2^m which have Γ_m as their translation-group, i.e. from C_2^1 or C_2^2.

The translations of a primitive triplet of Γ_0 are mutually orthogonal; $EE_1 = 2\tau_x$, $EE_2 = 2\tau_y$ (Fig. 86) are perpendicular to one another, and represent a primitive pair in their own plane. The arrangement of axes is therefore that shown in Fig. 90. All the axes of C_2^1 and C_2^2 are geometrically indistinguishable; hence it is sufficient to take the plane of $S(t)$ as either σ_0 or σ_m.

First consider the result of combining C_2^1 with a reflexion S_0 in σ_0. The group so obtained has a subgroup C_s^1 whose

planes are parallel to σ_0; for Γ_o may be considered as a particular case of a group Γ_m whose 2-al symmetry-axes are parallel to EG. Since the group $\{C_2^1, S_0\}$ contains a reflexion in the plane σ_0 passing through the 2-al rotation-axis e, it contains a reflexion in the plane σ_1 which is at right angles to σ_0 and also passes through e. The group therefore contains a subgroup C_s^1 whose planes are parallel to σ_1. Hence we have

$$C_{2v}^1 = \{C_2^1, S_0\} = \{C_2^1, S_1\}.$$

Subgroups* C_s^1, C_s^1.

This and the following groups are shown diagrammatically in Figs. 92 to 101.

An operation $A_{\pi,t}$ about e, combined with a reflexion S_0 in σ_0, is readily seen to be equivalent to an operation $S_I(t)$ in σ_1. Hence we have

$$C_{2v}^2 = \{C_2^2, S_0\} = \{C_2^2, S_1(\tau_z)\}.$$

Subgroups C_s^1, C_s^2.

If now the reduced operation in the plane σ_0 be not S_0 but $S_0(t)$ where $t \neq 0$, t must be τ_z, τ_x, or $\tau_x + \tau_z$, for the primitive pair of translations parallel to the plane containing the axes e and f is $2\tau_x, 2\tau_z$.

Consider, then, the group $\{C_2^1, S_0(\tau_z)\}$. The operation $A_{\pi,0}$ about e combined with $S_0(\tau_z)$ in σ_0 is readily seen to be equivalent to an operation $S_1(\tau_z)$ in σ_1. Hence we have

$$C_{2v}^3 = \{C_2^1, S_0(\tau_z)\} = \{C_2^1, S_1(\tau_z)\}.$$

Subgroups C_s^2, C_s^2.

$\{C_2^2, S_0(\tau_z)\} = \{C_2^2, S_1\}$ is not geometrically distinguishable from C_{2v}^2.

Again $A_{\pi,0}$ about e followed by $S_0(\tau_x)$ in σ_0 is equivalent to a reflexion S_1 in σ_1 followed by a translation τ_x; i.e. to a reflexion S_{m_1} in σ_{m_1}. Therefore we have

$$C_{2v}^4 = \{C_2^1, S_0(\tau_x)\} = \{C_2^1, S_{m_1}\}.$$

Subgroups C_s^2, C_s^1.

Similarly we have

$$C_{2v}^5 = \{C_2^2, S_0(\tau_x)\} = \{C_2^2, S_{m_1}(\tau_z)\}.$$

Subgroups C_s^2, C_s^2.

* First is given the subgroup whose planes are parallel to σ_0, then that whose planes are parallel to σ_1.

$C_{2v}^{6} = \{C_2^1, S_0(\tau_x+\tau_z)\} = \{C_2^1, S_{m_1}(\tau_z)\}$.
Subgroups C_s^2, C_s^2.

$C_{2v}^{7} = \{C_2^2, S_0(\tau_x+\tau_z)\} = \{C_2^2, S_{m_1}\}$.
Subgroups C_s^2, C_s^1.

$C_{2v}^{8} = \{C_2^1, S_m(\tau_x)\} = \{C_2^1, S_{m_1}(\tau_y)\}$.
Subgroups C_s^2, C_s^2.

$C_{2v}^{9} = \{C_2^2, S_m(\tau_x)\} = \{C_2^2, S_{m_1}(\tau_y+\tau_z)\}$.
Subgroups C_s^2, C_s^2.

$C_{2v}^{10} = \{C_2^1, S_m(\tau_x+\tau_z)\} = \{C_2^1, S_m(\tau_y+\tau_z)\}$.
Subgroups C_s^2, C_s^2.

There are no more groups C_{2v}^{m} whose translation-group is Γ_o, for $\{C_2^1, S_m\} = \{C_2^1, S_1(\tau_y)\}$ is not geometrically distinguishable from C_{2v}^{4}, $\{C_2^2, S_m\} = \{C_2^2, S_1(\tau_y+\tau_z)\}$ from C_{2v}^{7} $\{C_2^2, S_m(\tau_x+\tau_z)\} = \{C_2^2, S_{m_1}(\tau_y)\}$ from C_{2v}^{9}.

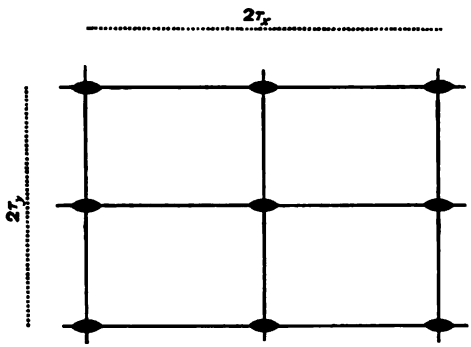

Fig. 92. C_{2v}^{1}.

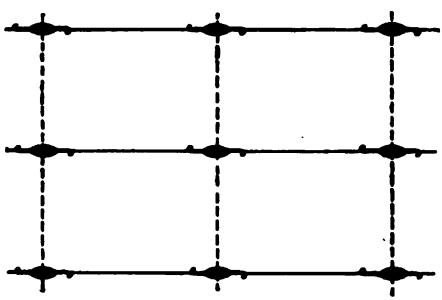

Fig. 93. C_{2v}^{2}.

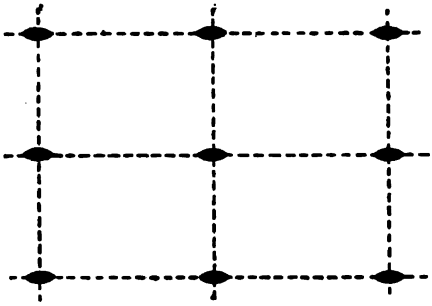

Fig. 94. C_{2v}^{3}.

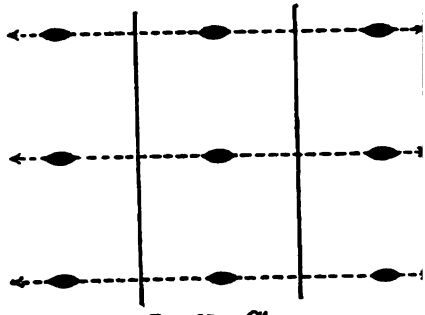

Fig. 95. C_{2v}^{4}.

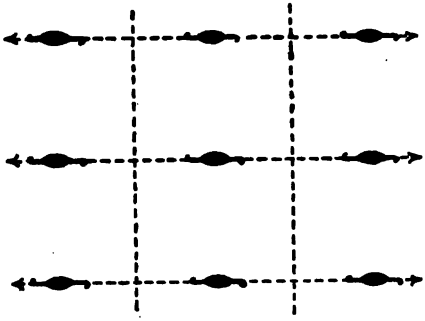

Fig. 96. C_{2v}^{5}.

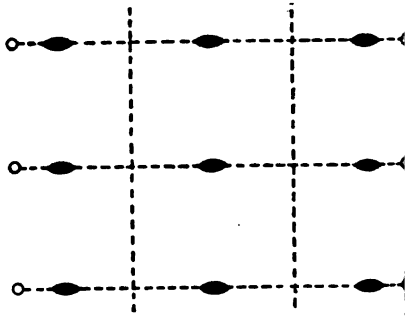

Fig. 97. C_{2v}^{6}.

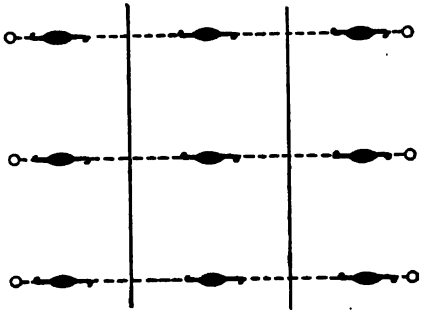

Fig. 98. C_{2v}^{7}.

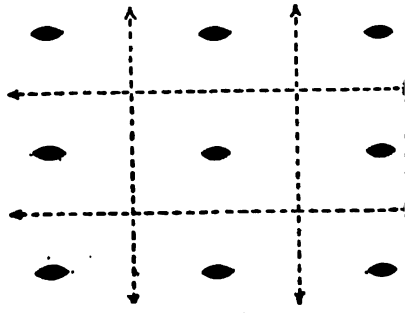

Fig. 99. C_{2v}^{8}.

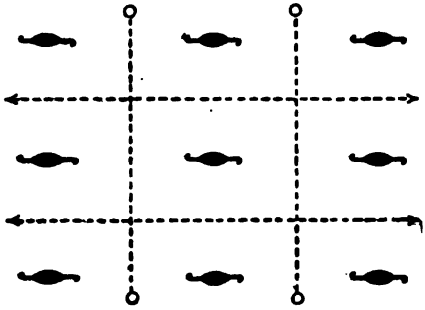

Fig. 100. C_{2v}^{9}.

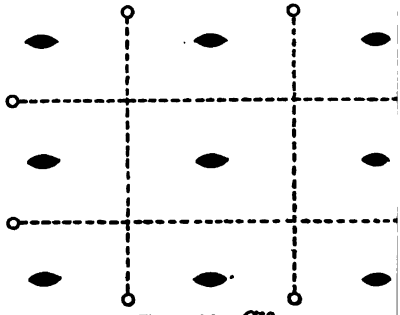

Fig. 101. C_{2v}^{10}.

§ 3. We shall now find the groups C_{2v}^m whose translation-group is Γ_o'.

First we shall suppose the axes of that group C_2^m from which C_{2v}^m is derived to be in the direction of τ_z*. Γ_o' is now a specialized form of Γ_m and therefore C_2^m must be either C_1^2 or C_2^2. Since a primitive triplet of Γ_o' is $2\tau_z$, $\tau_x+\tau_y$, $\tau_x-\tau_y$, the parallelogram $EE_1E_2E_3$ now takes the specialized form of Fig. 91 in which $EF = EG$; $EE_3 = 2\tau_x$, $E_1E_2 = 2\tau_y$.

If such a group C_{2v}^m contains an operation $S_d(t)$ in the plane σ_d it must contain a similar operation in a parallel plane through E_1, since EE_1 is a translation of the group. It contains, however, the operation $S_{d'}(t+\tau_x)$ in the plane $\sigma_{d'}$, for this is equivalent to $S_d(t)$ in σ_d followed by the translation $\tau_x-\tau_y$. In fact the subgroups C_s^m are now either C_s^3 or C_s^4.

Since no two axes of C_2^1 or of C_2^2 are geometrically distinguishable, it is sufficient to take σ_d as the plane of $S(t)$.

We have then (see Figures 102, 103, 104)

$$C_{2v}^{11} = \{C_2^1, S_d\} = \{C_2^1, S_{d_1}\}.$$

Subgroups † C_s^3, C_s^3.

$$C_{2v}^{12} = \{C_2^2, S_d\} = \{C_2^2, S_{d_1}(\tau_z)\}.$$

Subgroups C_s^3, C_s^4.

$$C_{2v}^{13} = \{C_2^1, S_d(\tau_z)\} = \{C_2^1, S_{d_1}(\tau_z)\}.$$

Subgroups C_s^4, C_s^4.

There are no more groups of this sort, for $\{C_2^2, S_d(\tau_z)\} = \{C_2^2, S_{d_1}\}$ is geometrically equivalent to C_{2v}^{12};

$$\{C_2^1, S_d(\tau_x+\tau_z)\} = \{C_2^1, S_{d'}(\tau_z)\} \text{ to } C_{2v}^{13}$$

(for the axes e, f are not geometrically distinguishable);

$$\{C_2^2, S_d(\tau_x+\tau_z)\} = \{C_2^2, S_{d'}(\tau_z)\} \text{ to } C_{2v}^{12}.$$

Now suppose the axes of C_2^m to be still in the direction of τ_z, but that Γ_o' is not expressed in the notation of p. 136, but is so orientated that $2\tau_x, \tau_y+\tau_z, \tau_y-\tau_z$ form a primitive triplet, instead of $2\tau_z, \tau_x+\tau_y, \tau_x-\tau_y$. In this case Γ_o' considered as a translation-group of C_2^m is a specialized form of

* The notation of p. 136 is used.

† First is given the subgroup whose planes are parallel to σ_d, then that whose planes are parallel to σ_{d_1}.

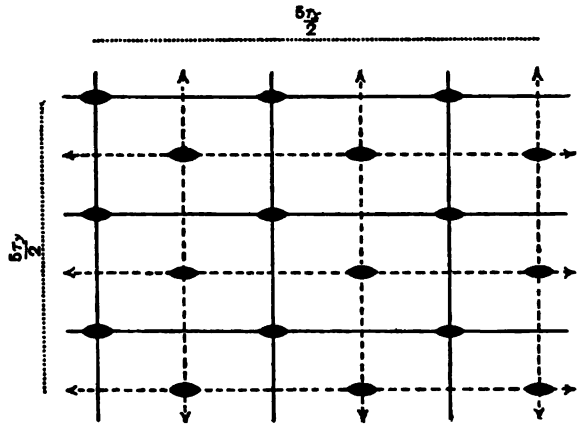

Fig. 102. C_{2v}^{11}.

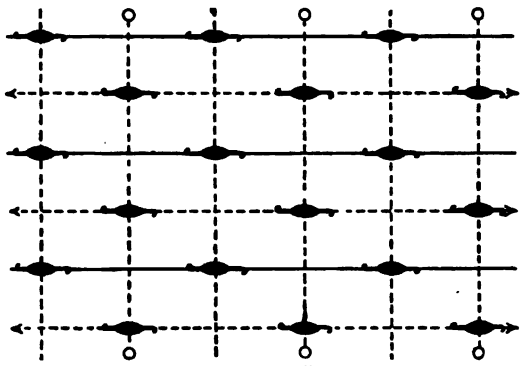

Fig. 103. C_{2v}^{12}.

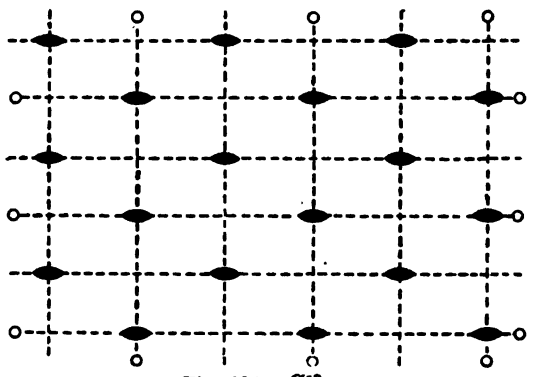

Fig. 104. C_{2v}^{13}.

Γ_m' (not of Γ_m), and therefore C_2^m must be C_2^3. (Fig. 105, in which $EE_1 = 2\tau_x$, $EE_2 = \tau_y$, $E_2R = UE_2 = \tau_z$).

The arrangement of axes is evidently that of Fig. 90; the

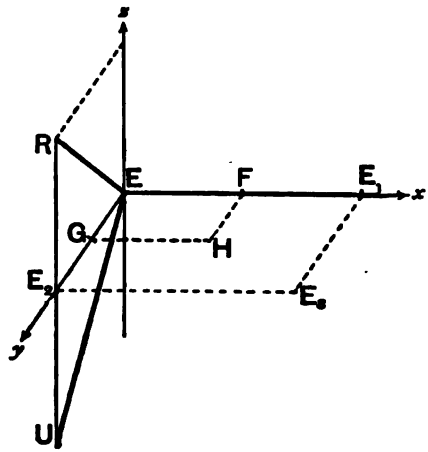

Fig. 105.

series **e**, **f** are rotation-axes; **g** and **h** are screw-axes.

Now the plane of $S(t)$ cannot be σ_m, for then this operation would bring screw-axes to coincide with rotation-axes, and therefore would not bring the system of axes of C_2^3 to self-coincidence.

A gliding-reflexion $S_0(t)$ in σ_0 involves a gliding-reflexion $S_0(t+\tau_z)$ in σ_0'*; it is therefore sufficient to take σ_0 as the plane of $S(t)$.

The subgroups whose planes are parallel to σ_0 must be either C_s^3 or C_s^4; those whose planes are parallel to σ_1 must be either C_s^1 or C_s^2.

We have (see Figures 106 to 109)

$$C_{2v}^{14} = \{C_2^3, S_0\} = \{C_2^3, S_1\}.$$

Subgroups C_s^3, C_s^1.

$$C_{2v}^{15} = \{C_2^3, S_0(\tau_z)\} = \{C_2^3, S_1(\tau_z)\}.$$

Subgroups C_s^3, C_s^2.

$$C_{2v}^{16} = \{C_2^3, S_0(\tau_x)\} = \{C_2^3, S_{m_1}\}.$$

Subgroups C_s^4, C_s^1.

$$C_{2v}^{17} = \{C_2^3, S_0(\tau_x+\tau_z)\} = \{C_2^3, S_{m_1}(\tau_z)\}.$$

Subgroups C_s^4, C_s^2.

* For $\tau_y + \tau_z$ is a translation of the group.

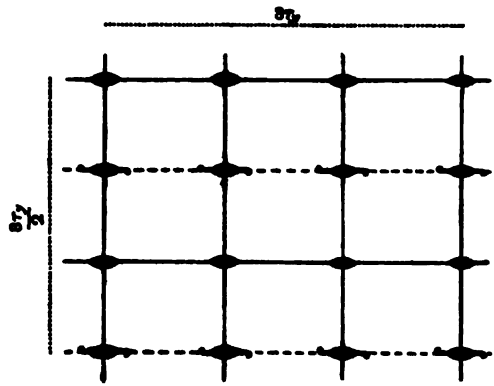

Fig. 106. C_{2v}^{14}.

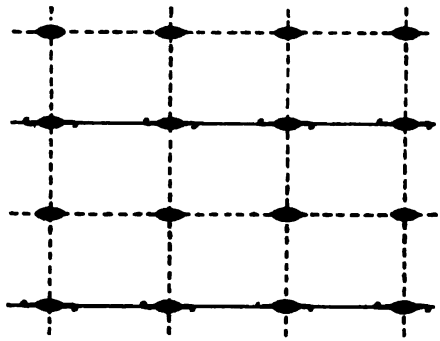

Fig. 107. C_{2v}^{15}.

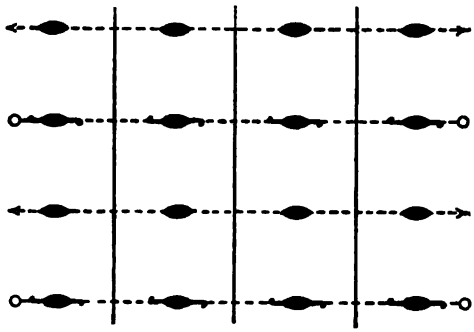

Fig. 108. C_{2v}^{16}.

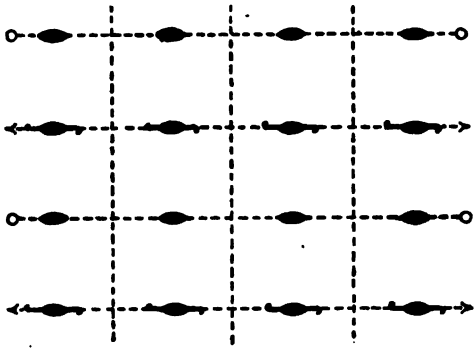

Fig. 109. C_{2v}^{17}.

§ 4. We now take Γ_o'' as translation-group; it is a specialized form of Γ_m', and therefore the subgroup C_2^m of C_{2v}^m must be C_2^3.

A primitive triplet is $\tau_y+\tau_z$, $\tau_z+\tau_x$, $\tau_x+\tau_y$, and hence the arrangement of axes is that of Fig. 90, where $EF = \frac{1}{2}\tau_x$, $EG = \frac{1}{2}\tau_y$ (compare also Fig. 110 where $EE' = 2\tau_z$). In this case the series of axes e, h are rotation-axes, f, g are screw-axes.

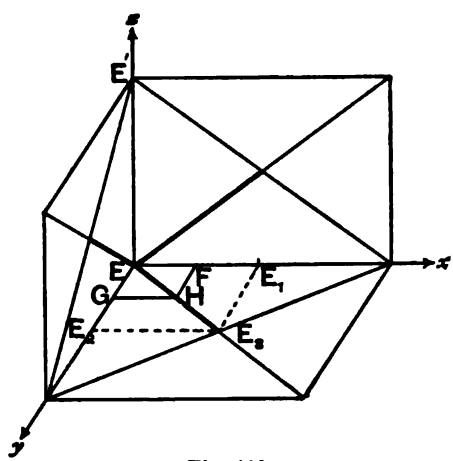

Fig. 110.

If such a group contains the operation $S(t)$ in a plane σ parallel to σ_0, it contains an operation of the form $S(t+\tau_z)$ in another parallel plane distant $\frac{1}{2}\tau_y$ from σ; it is only necessary therefore to consider operations $S(t)$ in σ_0 and σ_m. t must be o, $\frac{1}{2}(-\tau_x+\tau_z)$, $\frac{1}{2}(\tau_x+\tau_z)$, or τ_z, for $\tau_x+\tau_z$, $-\tau_x+\tau_z$ are a primitive pair of translations in the plane perpendicular to τ_y (cf. p. 184 (4)). Both subgroups C_s^m must be either C_s^3 or C_s^4.

We have then $C_{2v}^{18} = \{C_2^3, S_0\} = \{C_2^3, S_1\}$.

Subgroups C_s^3, C_s^3.

$C_{2v}^{19} = \{C_2^3, S_m(\frac{1}{2}\tau_x + \frac{1}{2}\tau_z)\} = \{C_2^3, S_m(\frac{1}{2}\tau_y + \frac{1}{2}\tau_z)\}$.
Subgroups C_s^4, C_s^4.

This exhausts the number of groups whose translation-group is Γ_o'', for $S_0(-\frac{1}{2}\tau_x + \frac{1}{2}\tau_z)$, $S_0(\frac{1}{2}\tau_x + \frac{1}{2}\tau_z)$, S_m, $S_m(\tau_z)$ do not bring the system of axes of C_2^3 into self-coincidence; while $\{C_2^3, S_0(\tau_z)\} = \{C_2^3, S_0'\}$ is not geometrically distinguishable from C_{2v}^{18}, for **e** and **h** on the one hand and **f** and **g** on the other cannot be distinguished; and

$$\{C_2^3, S_m(-\tfrac{1}{2}\tau_x + \tfrac{1}{2}\tau_z)\} = \{C_2^3, S_m(\tfrac{1}{2}\tau_x - \tfrac{1}{2}\tau_z)\} = \{C_2^3, S_m(\tfrac{1}{2}\tau_x + \tfrac{1}{2}\tau_z - \tau_z)\}$$

is seen in the same way to be identical with C_{2v}^{19}.

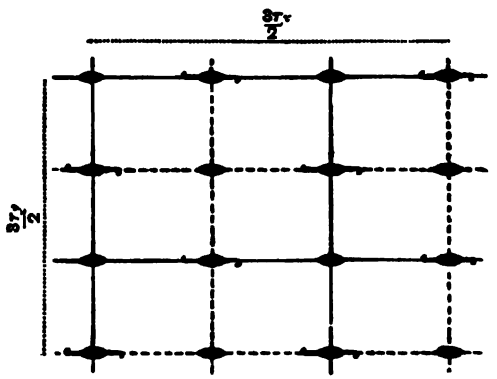

Fig. 111. C_{2v}^{18}.

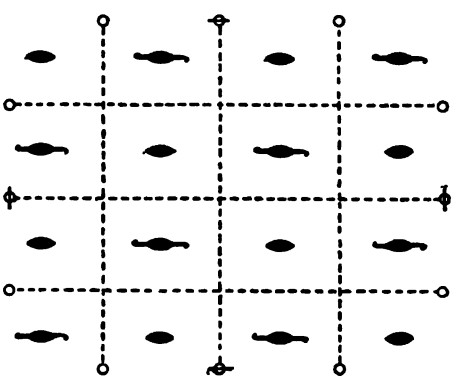

Fig. 112. C_{2v}^{19}.

In Fig. 111 the planes represented as corresponding to an operation of the form $S(\tau_z)$, might be equally well represented as corresponding to $S(\tau_x)$ for $\tau_y+\tau_z$ is a translation of the group.

In Fig. 112 the alternate planes in either direction have different translation-components; this is shown by the short lines through the small circles at the ends of the lines representing the alternate planes.

§ 5. Lastly, we consider groups C_{2v}^m whose translations are

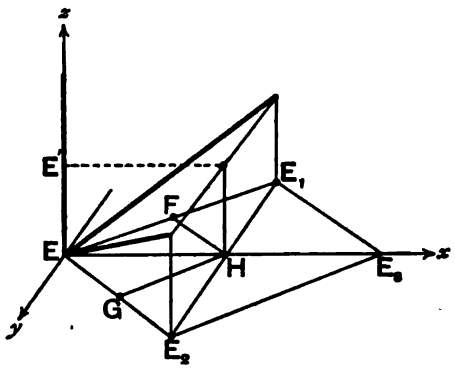

Fig. 113.

represented by Γ_o'''. Since this is a specialized form of Γ_m' the subgroup C_2^m is C_2^3.

We take as a primitive triplet of Γ_o'''

$$2\tau_z,\ \tau_x+\tau_y+\tau_z,\ \tau_x-\tau_y+\tau_z;$$

the arrangement of axes is then that of Fig. 91, where $EF=\frac{1}{2}(\tau_x-\tau_y)$, $EG=\frac{1}{2}(\tau_x+\tau_y)$. (See also Fig. 113, where $EE'=\tau_z$). In this case **e** and **h** are rotation-axes, **f** and **g** are screw-axes.

If such a group contains an operation $S(t)$ in a plane σ parallel to σ_d, it contains an operation $S(t+\tau_x+\tau_z)$ in another parallel plane distant $\frac{1}{2}\tau_y$ from σ; it is only necessary therefore to consider operations $S(t)$ in σ_d.

t must be 0, τ_x, τ_z, or $\tau_x+\tau_z$, for $2\tau_x$, $2\tau_z$ are a primitive pair of translations in the plane perpendicular to τ_y*.

Both subgroups C_s^m must be C_s^3 or C_s^4.

* It may be verified from the general condition of p. 124, or otherwise, that the two translations $2\tau_z$, $(\tau_x+\tau_y+\tau_z)+(\tau_x-\tau_y+\tau_z)-2\tau_z$ form a primitive pair, if $\tau_x+\tau_y+\tau_z$, $\tau_x-\tau_y+\tau_z$, $2\tau_z$ are a primitive triplet.

We have

$$C_{2v}^{20} = \{C_2^3, S_d\} = \{C_2^3, S_{d_1}\}.$$

Subgroups C_s^3, C_s^3.

$$C_{2v}^{21} = \{C_2^3, S_d(\tau_z)\} = \{C_2^3, S_{d_1}(\tau_z)\}.$$

Subgroups C_s^4, C_s^4.

$$C_{2v}^{22} = \{C_2^3, S_d(\tau_x)\} = \{C_2^3, S_{d_1}\} = \{C_2^3, S_{d_1}(\tau_y+\tau_z)\}.$$

Subgroups C_s^4, C_s^3.

$\{C_2^3, S_d(\tau_x+\tau_z)\}$ is not distinguishable from $\{C_2^3, S_{d_1}(\tau_y+\tau_z)\}$, i.e. from C_{2v}^{22}.

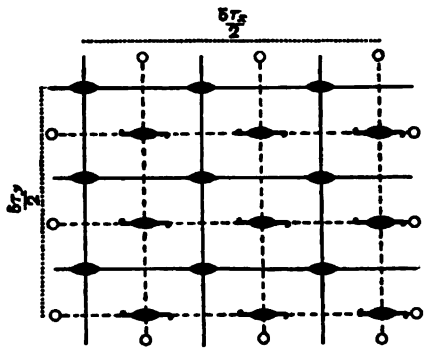

Fig. 114. C_{2v}^{20}.

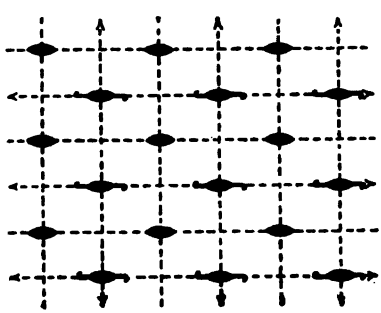

Fig. 115. C_{2v}^{21}.

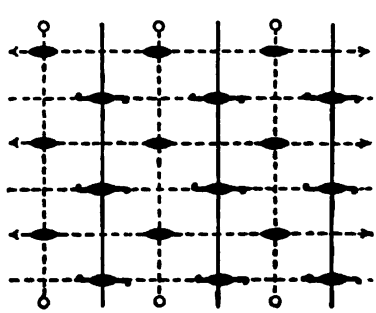

Fig. 116. C_{2v}^{22}.

§ 6. Orthorhombic Enantiomorphy.

Since the group $D_2(=Q)$ can be derived from C_2 by multiplying the operations of C_2 by a rotation through π about a line perpendicular to the axis of C_2, therefore all groups

$D_2^m (= Q^m)$ can be derived from C_2^m by multiplying the operations of C_2^m by an operation $A_{\pi,t}$ about an axis a perpendicular to the axes of C_2^m; where $A_{\pi,t}$ brings the system of axes and translations of C_2^m into self-coincidence, and is such that $(A_{\pi,t})^2$ is a translation of C_2^m. Such an investigation is exactly parallel to that of §§ 2–5, but is rather simpler, as it is only necessary to consider the case of $t = 0$, and $t =$ half a translation in the direction of a. It will, however, be more convenient for our present purpose to choose another method.

Since Q $(= D_2)$ has three subgroups C_2 whose axes are mutually orthogonal, therefore $Q^m (= D_2^m)$ has three subgroups C_2^m such that the axes of any one of them are at right angles to the axes of both the others. We shall denote these subgroups by U, V, W; their axes are parallel to the symmetry-axes of the lattice representing the translation-group (p. 159). We use the notation of pp. 135, 136 and suppose the primitive translations of the lattice in the directions of the axes of U, V, W to be $2\tau_x$, $2\tau_y$, $2\tau_z$ respectively.

If a is an axis of one of the subgroups U, V, W, the screw about a brings the system of axes of each of the other two to self-coincidence; hence evidently a must meet an axis, or pass halfway between two axes, of each of the other two subgroups.

§ 7. If a group Q^m has the translation-group Γ_0 (primitive triplet $2\tau_x$, $2\tau_y$, $2\tau_z$), all its subgroups C_2^m must be either C_1^2 or C_2^2. There are four possibilities therefore, the subgroups may be C_2^1, C_2^1, C_2^1; C_2^1, C_2^1, C_2^2; C_2^2, C_2^2, C_2^1; or C_2^2, C_2^2, C_2^2.

If the subgroups U, V, W are all C_2^1 their axes are all rotation-axes. Any axis of U must meet an axis of V; for were it not so W must contain screw-axes, since the resultant of two rotations through π about non-intersecting axes is a screw (p. 153 (2)), which is contrary to the hypothesis that W has only rotation-axes.

Through the point of intersection of two rotation-axes a third must pass; hence the arrangement of axes must be that of Fig. 117*.

Q^1. Subgroups C_2^1, C_2^1, C_2^1.

Again, if two of the subgroups, e. g. U and V, are C_2^1, but the third is C_2^2; the axes of U, V are all rotation-axes and those of W are all screw-axes. No axis of U can meet an

* The explanation of these figures is given on p. 190.

axis of V, but the line perpendicular to and meeting a rotation-axis of U and a rotation-axis of V must be a screw-axis of W. Any rotation-axis of the group meets screw-axes and passes halfway between rotation-axes whose direction is perpendicular to it (Fig. 118).

Q^2. Subgroups C_2^1, C_2^1, C_2^2.

If U and V are C_2^2 and W is C_2^1 the axes of U and V are all screw-axes, those of W all rotation-axes. No axis of W can cut an axis of U or V, for else the group would have rotation-axes whose direction is that of the axes of V or U respectively (p. 154 (4)). On the other hand the axes of U and V cut, for otherwise W would contain screw-axes. Therefore the arrangement of axes must be that of Fig. 119.

Q^3. Subgroups C_2^2, C_2^2, C_2^1.

If U, V, W are all C_2^2, all the axes of the group are screw-axes and therefore (p. 153 (3)) no two axes cut. This leads of necessity to the arrangement of Fig. 120.

Q^4. Subgroups C_2^2, C_2^2, C_2^2.

In Figs. 117 to 125 the arrangement of the axes of the groups Q^1 to Q^9 is shown.

Rotation-axes are denoted by thick lines, screw-axes by thick broken lines, and the sides of a rectangular parallelepiped of sides τ_x, τ_y, τ_z by ordinary lines*. The complete series of axes of any one of these groups would be given by filling space with repetitions of the corresponding figure, arranged in the same way as the parallelepipeda of Fig. 66.

§ 8. If the translation-group is Γ_o' (primitive triplet $\tau_x+\tau_y$, $\tau_x-\tau_y$, $2\tau_z$) U and V must be C_2^3, and W is either C_1^2 or C_2^2 (cp. § 3, pp. 181 to 183).

The arrangement of the axes of W is that of Fig. 91; and since the operations of U and V bring this system of axes into self-coincidence, the axes of U and V must be parallel to the diagonals of the rhombus of that figure.

In the group whose subgroups are C_2^3, C_2^3, C_2^2 the axes of W are all screw-axes, and therefore no two rotation-axes of U and V can meet. Since, however, a rotation about one of the rotation-axes of U, followed by a rotation about any one of the rotation-axes of V, is equivalent to a screw about a line meeting both axes, therefore the screw-axes of W cut

* Except in the case of Fig. 123, when the sides of the parallelepiped are $\frac{1}{2}\tau_x$, $\frac{1}{2}\tau_y$, $\frac{1}{2}\tau_z$.

rotation-axes of U and V. These considerations show that the arrangement of axes must be that of Fig. 121.

Q^5. Subgroups C_2^3, C_2^3, C_2^2.

If however the axes of W are rotation-axes, the rotation-axes of U and V must intersect, and axes of W must pass through their points of intersection; this leads at once to Fig. 122.

Q^6. Subgroups C_2^3, C_2^3, C_2^1.

§ 9. If the translation-group is Γ_o'' (primitive triplet $\tau_y+\tau_z$, $\tau_z+\tau_x$, $\tau_x+\tau_y$) U, V, W are all C_2^3.

The arrangement of the axes of U, V, and W is that shown in Fig. 90, where **e**, **h** are rotation-axes and **f**, **g** screw-axes (p. 185).

Since a rotation through π about any rotation-axis of U brings the system of axes of W into self-coincidence, therefore any rotation-axis of U must meet rotation-axes *and* screw-axes of W. Through the points where it meets the former, rotation-axes of V must pass; through the points where it meets the latter axes of V pass which cannot be rotation-axes, and which are therefore screw-axes. These considerations with the help of Fig. 90 lead at once to the arrangement of Fig. 123.

Q^7. Subgroups C_2^3, C_2^3, C_2^3.

In Fig. 123 the sides of the parallelepiped are $\frac{1}{2}\tau_x$, $\frac{1}{2}\tau_y$, $\frac{1}{2}\tau_z$ not τ_x, τ_y, τ_z as in the other cases.

§ 10. If the translation-group is Γ_o''' (for which $\tau_x+\tau_y+\tau_z$, and any two of $2\tau_x$, $2\tau_y$, $2\tau_z$ form a primitive triplet) U, V, W are all C_2^3. The arrangement of the axes of U, V, and W is that of Fig. 91, where **e**, **h** are rotation-axes and **f**, **g** screw-axes (p. 187). A rotation-axis of U must pass either through two rotation-axes or two screw-axes of W*. In the former case rotation-axes of V pass through the intersection of rotation-axes of U and W, which consideration with the help of Fig. 91 leads at once to Fig. 124.

Q^8. Subgroups C_2^3, C_2^3, C_2^3.

In the latter case a rotation-axis of U cannot meet a rotation-axis of V; but it meets screw-axes of V, which intersect and are perpendicular to rotation-axes of U and W. The arrangement of axes is easily seen to be that of Fig. 125.

* This is evident from an inspection of Fig. 91, since the rotation about this axis of U must bring the system of axes of W into self-coincidence.

§ 11. The reader will find it a great assistance to the understanding of what follows if he will make himself thoroughly familiar with Figs. 117 to 125.

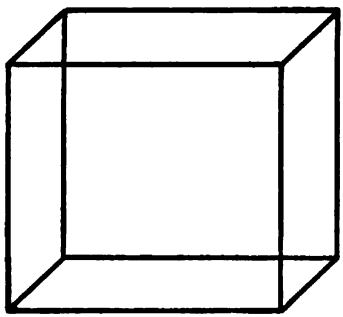

Fig. 117. Q^1.

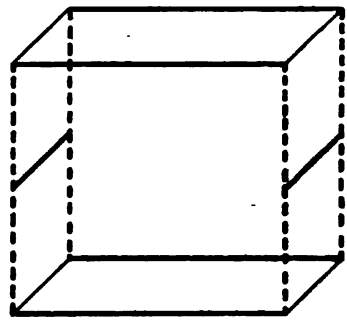

Fig. 118. Q^2.

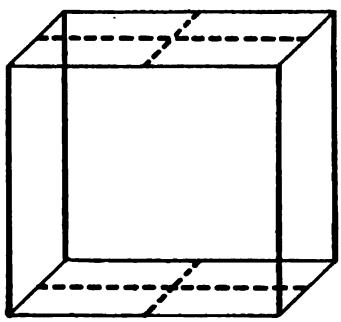

Fig. 119. Q^3.

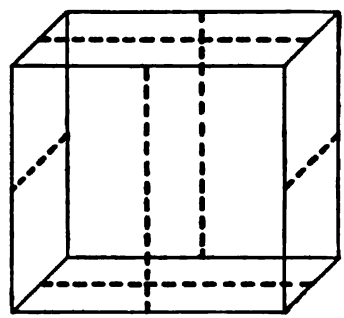

Fig. 120. Q^4.

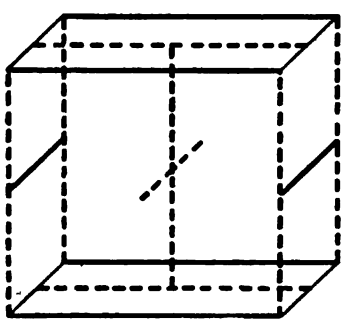

Fig. 121. Q^5.

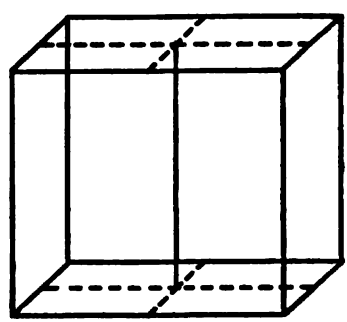

Fig. 122. Q^6.

The above method has the advantage that it shows most

readily the fact that no group Q^m can have its axes arranged in any way other than those shown in Figs. 117 to 125. It is not quite so clear perhaps that to each of these arrangements a group really corresponds. This may be shown as follows.

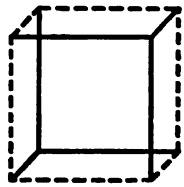

Fig. 123. Q^7.

Let A be the operation of U about any one (a) of the axes of U as shown in any one of these figures. Then in each case A brings the system of axes of W to self-coincidence, and A^2 is a translation of the group. Therefore, by p. 164, $\{W, A\}$ is really a group; and in each figure the

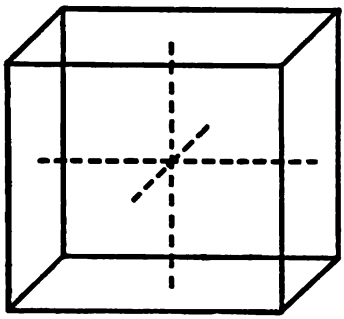

Fig. 124. Q^8.

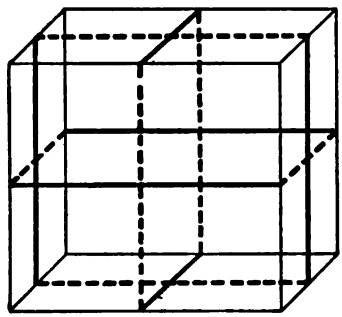

Fig. 125. Q^9.

axes of U other than a and the axes of V are a necessary consequence of the coexistence of a and the axes of W.

§ 12. Orthorhombic Holohedry.

Since the point-group Q_h is obtained by multiplying the operations of Q by an inversion, therefore the groups Q_h^m are derived from such groups Q^m as satisfy the condition of p. 165, that is, from *all* groups Q^m, by multiplying by an inversion which brings the system of axes of Q^m into self-coincidence. Since an inversion about a point P followed by an inversion about a point P' is equivalent to a translation $2PP'$, therefore two groups Q_h^m formed from the same group Q^m by multiplying by inversions about the points P, P' are only identical if PP' is half a translation of the group, or if P and P' are similarly situated with regard to the axes of Q^m. Inversion about a centre of symmetry of Q_h^m must bring the system of axes of Q^m into self-coincidence, and therefore every centre of symmetry must lie in a vertex,

the middle, the middle of an edge, or the middle of a face of a rectangular parallelepiped p, such as is shown in Figs. 117 to 125.

We shall denote the centre of p by R, the middle points of its edges by L, M, N, &c.; the middle points of its faces by D, E, F, &c.; and its vertices by A, A_1, &c. (Fig. 126).

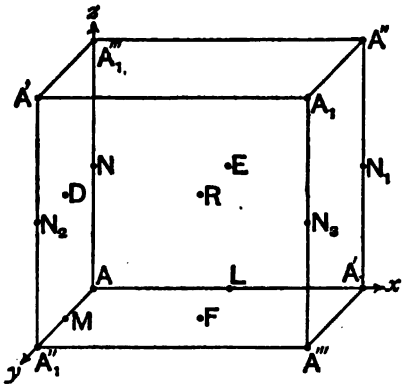

Fig. 126.

Unless p is the parallelepiped of Fig. 123, a group which has a centre of symmetry at one of the vertices of p has centres of symmetry at all the other vertices; for each edge of p represents half a translation of the group.

Since Q_h has three subgroups C_{2h} and three subgroups C_{2v}, Q_h^m has three subgroups C_{2h}^m and three subgroups C_{2v}^m. These will be given for each group; that subgroup which has its axes in the direction of τ_x being put first, then that which has its axes in the direction of τ_y, and lastly, that which has its axes in the direction of τ_z. A reference to the figures given for the groups Q^m and C_{2v}^m will be sufficient to show completely the arrangement of all the axes and planes of each group Q_h^m. Every plane of symmetry or gliding-plane is perpendicular to one of the three series of axes and parallel to the other two.

§ 13. First consider the groups whose translation-group is Γ_0, of which $2\tau_x, 2\tau_y, 2\tau_z$ form a primitive triplet.

Starting from Q^1 we have *

$$Q_h^1 = \{Q^1, I\}.$$
Subgroups $C_{2h}^1, C_{2h}^1, C_{2h}^1$; $C_{2v}^1, C_{2v}^1, C_{2v}^1$.

$$Q_h^2 = \{Q^1, I_\tau\}.$$
Subgroups $C_{2h}^4, C_{2h}^4, C_{2h}^4$; $C_{2v}^{10}, C_{2v}^{10}, C_{2v}^{10}$.

L, M, N have similar situations with regard to the axes of Q^1, the same is true of D, E, F; hence the only other groups

* Inversions about A, R, L, M, N, D, E, F are denoted by $I, I_r, I_l, I_m, I_n, I_d, I_e, I_f$ respectively.

Q_h^m derivable from Q^1 are

$$Q_h^3 = \{Q^1, I_n\}.$$
Subgroups $C_{2h}^4, C_{2h}^4, C_{2h}^1; C_{2v}^4, C_{2v}^4, C_{2v}^3$.
$$Q_h^4 = \{Q^1, I_f\}.$$
Subgroups $C_{2h}^4, C_{2h}^4, C_{2h}^2; C_{2v}^5, C_{2v}^5, C_{2v}^3$.

R and F have similar situations with regard to the axes of Q^2, and so have A and N, L and D, M and E; hence the only distinct groups derivable from Q^2 are

$$Q_h^5 = \{Q^2, I\}.$$
Subgroups $C_{2h}^1, C_{2h}^4, C_{2h}^2; C_{2v}^4, C_{2v}^1, C_{2v}^3$.
$$Q_h^6 = \{Q^2, I_r\}.$$
Subgroups $C_{2h}^4, C_{2h}^4, C_{2h}^5; C_{2v}^3, C_{2v}^{10}, C_{2v}^9$.
$$Q_h^7 = \{Q^2, I_l\}.$$
Subgroups $C_{2h}^1, C_{2h}^4, C_{2h}^5; C_{2v}^3, C_{2v}^4, C_{2v}^7$.
$$Q_h^8 = \{Q^2, I_m\}.$$
Subgroups $C_{2h}^4, C_{2h}^4, C_{2h}^5; C_{2v}^3, C_{2v}^3, C_{2v}^5$.

L and M, D and E have similar situations with regard to the axes of Q^3; the distinct groups derivable from Q^3 are

$$Q_h^9 = \{Q^3, I\}.$$
Subgroups $C_{2h}^5, C_{2h}^5, C_{2h}^1; C_{2v}^2, C_{2v}^2, C_{2v}^3$.
$$Q_h^{10} = \{Q^3, I_r\}.$$
Subgroups $C_{2h}^5, C_{2h}^5, C_{2h}^4; C_{2v}^9, C_{2v}^9, C_{2v}^3$.
$$Q_h^{11} = \{Q^3, I_m\}.$$
Subgroups $C_{2h}^2, C_{2h}^5, C_{2h}^4; C_{2v}^5, C_{2v}^2, C_{2v}^4$.
$$Q_h^{12} = \{Q^3, I_n\}.$$
Subgroups $C_{2h}^5, C_{2h}^5, C_{2h}^1; C_{2v}^7, C_{2v}^7, C_{2v}^{10}$.
$$Q_h^{13} = \{Q^3, I_f\}.$$
Subgroups $C_{2h}^2, C_{2h}^2, C_{2h}^4; C_{2v}^7, C_{2v}^7, C_{2v}^1$.
$$Q_h^{14} = \{Q^3, I_d\}.$$
Subgroups $C_{2h}^5, C_{2h}^5, C_{2h}^4; C_{2v}^9, C_{2v}^5, C_{2v}^5$.

A and R have similar situations with regard to the axes of Q^4 and so have L, M, N, D, E, F; the distinct groups derivable from Q^4 are therefore

$$Q_h^{15} = \{Q^4, I\}.$$
Subgroups $C_{2h}^5, C_{2h}^5, C_{2h}^5; C_{2v}^5, C_{2v}^5, C_{2v}^5$.
$$Q_h^{16} = \{Q^4, I_s\}.$$
Subgroups $C_{2h}^5, C_{2h}^5, C_{2h}^2; C_{2v}^2, C_{2v}^7, C_{2v}^9$.

§ 14. Now let the translation-group be Γ_o', of which $2\tau_z$, $\tau_x+\tau_y$, $\tau_x-\tau_y$ is a primitive triplet. AF, NR, LM, DE are now half translations, so that if a group Q_h^m whose translation-group is Γ_o' has a centre of symmetry at A it has one at F, and if it has a centre at R it has one at N, &c.

Again, A and N, L and D are similarly situated with regard to the axes of Q^5, but not with regard to the axes of Q^6; therefore the distinct groups derivable from Q^5 are

$$Q_h^{17} = \{Q^5, I\}.$$
Subgroups $C_{2h}^3, C_{2h}^6, C_{2h}^2$; $C_{2v}^{16}, C_{2v}^{14}, C_{2v}^{12}$.
$$Q_h^{18} = \{Q^5, I_l\}.$$
Subgroups $C_{2h}^3, C_{2h}^6, C_{2h}^5$; $C_{2v}^{17}, C_{2v}^{15}, C_{2v}^{12}$.

And those derivable from Q^6 are

$$Q_h^{19} = \{Q^6, I\}.$$
Subgroups $C_{2h}^3, C_{2h}^3, C_{2h}^1$; $C_{2v}^{14}, C_{2v}^{14}, C_{2v}^{11}$.
$$Q_h^{20} = \{Q^6, I_r\}.$$
Subgroups $C_{2h}^6, C_{2h}^6, C_{2h}^1$; $C_{2v}^{16}, C_{2v}^{16}, C_{2v}^{13}$.
$$Q_h^{21} = \{Q^6, I_l\}.$$
Subgroups $C_{2h}^3, C_{2h}^3, C_{2h}^4$; $C_{2v}^{15}, C_{2v}^{15}, C_{2v}^{11}$.
$$Q_h^{22} = \{Q^6, I_d\}.$$
Subgroups $C_{2h}^6, C_{2h}^3, C_{2h}^4$; $C_{2v}^{17}, C_{2v}^{17}, C_{2v}^{13}$.

§ 15. Now let the translation-group be Γ_o''' of which a primitive triplet is $\tau_y+\tau_z$, $\tau_z+\tau_x$, $\tau_x+\tau_y$. If A is a centre of symmetry, A', A'', A''' are also centres of symmetry, but A_1, A_1', A_1'', A_1''' are not *.

However the group formed by combining an inversion about A with the operations of Q^7 (which is the only group Q^m whose translation-group is Γ_o'') is the same as that formed by combining an inversion about A_1, for A and A_1 have similar situations with regard to the axes of Q^7. Now the only other kind of inversion which brings the system of axes of Q^7 into self-coincidence is that about R (Fig. 126); hence the only two groups of this sort are

$$Q_h^{23} = \{Q^7, I\}.$$
Subgroups $C_{2h}^3, C_{2h}^3, C_{2h}^3$; $C_{2v}^{18}, C_{2v}^{18}, C_{2v}^{18}$.
$$Q_h^{24} = \{Q^7, I_r\}.$$
Subgroups $C_{2h}^6, C_{2h}^6, C_{2h}^6$; $C_{2v}^{19}, C_{2v}^{19}, C_{2v}^{19}$.

* Note that the sides of the parallelepiped of Fig. 123 are only $\frac{1}{2}\tau_z$, $\frac{1}{2}\tau_y$, $\frac{1}{2}\tau_x$.

§ 16. Lastly, we consider the translation-group Γ_o'''. This contains the translation $\tau_x+\tau_y+\tau_z$ so that a group possessing an inversion about A has one about R; similarly a group with an inversion about L, M, or N has one about D, E, or F respectively. Again L, M, N have similar situations with regard to the axes both of Q^8 and Q^9; hence we have only

$$Q_h^{25} = \{Q^8, I\}.$$

Subgroups $C_{2h}^3, C_{2h}^3, C_{2h}^3$; $C_{2v}^{20}, C_{2v}^{20}, C_{2v}^{20}$.

$$Q_h^{26} = \{Q^8, I_n\}.$$

Subgroups $C_{2h}^6, C_{2h}^6, C_{2h}^3$; $C_{2v}^{22}, C_{2v}^{22}, C_{2v}^{21}$.

$$Q_h^{27} = \{Q^9, I\}.$$

Subgroups $C_{2h}^6, C_{2h}^6, C_{2h}^6$; $C_{2v}^{21}, C_{2v}^{21}, C_{2v}^{21}$.

$$Q_h^{28} = \{Q^9, I_n\}.$$

Subgroups $C_{2h}^3, C_{2h}^3, C_{2h}^6$; $C_{2v}^{22}, C_{2v}^{22}, C_{2v}^{20}$.

CHAPTER XX

TETRAGONAL GROUPS.

§ 1. The translation-group of any one of these groups is either Γ_t (primitive triplet, $2\tau_x$, $2\tau_y$, $2\tau_z$) or Γ_t' (primitive triplet, $\tau_y+\tau_z$, $\tau_z+\tau_x$, $\tau_x+\tau_y$).

In either case τ_x and τ_y are equal in magnitude, and the 4-al axes of the groups are in the direction of τ_z.

Any plane perpendicular to τ_z will be called a 'principal' plane.

§ 2. Tetragonal Tetartohedry of the Second Sort.

The point-group C_4' is obtained by multiplying the operations of C_2 by a rotatory-reflexion of angle $\frac{\pi}{2}$. Hence all space-groups $C_4'^m$ are obtained by combining the operations of a group C_2^m with an operation A', which must bring the system of axes of C_2^m to self-coincidence, be isomorphous with a rotatory-reflexion of angle $\frac{\pi}{2}$, and be such that A'^2 is an operation of C_2^m (p. 164). These conditions are satisfied if and only if A' is a rotatory-reflexion of angle $\frac{\pi}{2}$ (see p. 154), whose axis coincides with a rotation-axis of C_2^m. C_2^2 has no rotation-axis; therefore we have evidently the two groups

$$C_4'^1 = \{C_2^1, A'\}$$
$$C_4'^2 = \{C_2^3, A'\}$$

of this class, and these two only.

The arrangement of their axes is shown in Figs. 127 and 128, in which ◆ denotes one of a series of 4-al axes whose planes coincide, and ◈ one of a series whose planes are at a distance $\frac{\tau_z}{2}$ from those of the other series.

Since A' followed by the translation $2m\tau_z$ is equivalent to a rotatory-reflexion, whose axis is the axis of A' and whose plane is at a distance $m\tau_z$ from the plane of A', these groups

contain an indefinite number of rotatory-reflexions about the same axis. The corresponding planes of rotatory-reflexion form a set of interval τ_s.

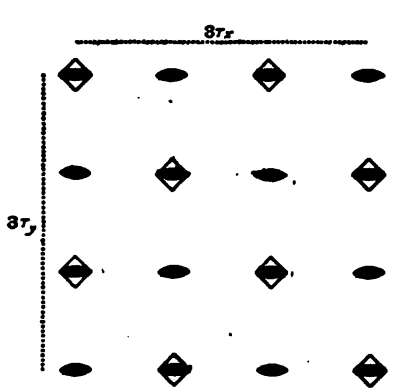

Fig. 127. C^1_4.

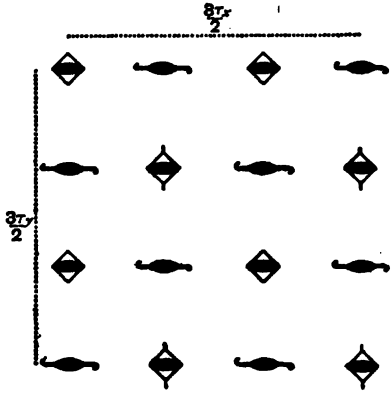

Fig. 128. C^2_4.

§ 8. Tetragonal Hemihedry of the Second Sort.

The group D_{2d} is derived from D_2 by multiplying by a reflexion in a plane parallel to one axis (w) and bisecting the angle between the other two; therefore groups D^m_{2d} are derived from D_2^m (or Q^m) by combining with an operation $S(t)$ in a plane parallel to one set of axes and bisecting the angle between the other two sets which brings the system of axes to self-coincidence. In order that this last condition may be fulfilled the plane of $S(t)$ must evidently be parallel to one of the planes *NRFA*, *DELM* of Fig. 126*; we shall take it parallel to *NRFA* †.

Again, since the axis w is a 4-al axis of the second sort for D_{2d}, therefore D^m_{2d} contains operations isomorphous with rotatory-reflexions of angle $\frac{\pi}{2}$; that is, D^m_{2d} has 4-al axes of the second sort. These must be 2-al rotation-axes of the group Q^m from which D^m_{2d} is derived; hence no groups D^m_{2d} can be derived from Q^2, Q^4, and Q^5 which have no rotation-axes parallel to AN (Fig. 126). The position of the 4-al axes of the second sort may be readily found in each case. Suppose

* This is so in the case of Q^2, Q^3, Q^5 and Q^6; in the other groups Q^m we may still take the plane of $S(t)$ parallel to *NRFA*, since the arrangement of all three sets of axes is the same.

† Of course, if a group D^m_{2d} has an operation $S(t)$ in a plane parallel to *DELM* it has a similar operation in a plane parallel to *NRFA*.

CHAPTER XX

TETRAGONAL GROUPS.

§ 1. The translation-group of any one of these groups is either Γ_t (primitive triplet, $2\tau_x$, $2\tau_y$, $2\tau_z$) or Γ_t' (primitive triplet, $\tau_y+\tau_z$, $\tau_z+\tau_x$, $\tau_x+\tau_y$).

In either case τ_z and τ_y are equal in magnitude, and the screw-axes of the groups are in the direction of τ_z.

Any plane perpendicular to τ_z will be called a 'principal plane.'

§ 2. TETRAGONAL TETARTOHEDRY OF THE SECOND SORT.

The point-group C_4' is obtained by multiplying the operations of C_2 by a rotatory-reflexion of angle $\frac{\pi}{2}$. Hence all space-groups C_4^m are obtained by combining the operations of a group Γ with an operation A', which must bring the system of Γ of C_4^m to self-coincidence, be isomorphous with a rotatory-reflexion of angle $\frac{\pi}{2}$, and be such that A'^2 is an operation of Γ (p. 164). These conditions are satisfied if and only if A' is a rotatory-reflexion of angle $\frac{\pi}{2}$ (see p. 154), whose axis coincides with a rotation-axis of C_2^m. C_2^2 has no rotation-axis, therefore we have evidently the two groups

$$C_4^1 = \{C_2^1, A'\}$$
$$C_4^2 = \{C_2^2, A'\}$$

of this class, and these two only.

The arrangement of their axes is shown in Figs. 127 and 128, in which ⬌ denotes one of a series of [illegible] axes whose planes coincide, and ⬍ one of a [illegible] at a distance $\frac{1}{2}$ from those of the [illegible]

Since A' followed by the [illegible] a rotatory-reflexion, whose [illegible] plane is at a distance [illegible]

an indefinite number of rotatory-reflexions about the axis. The corresponding planes of rotatory-reflexion set of interval τ_z.

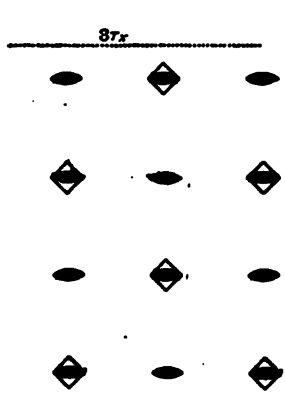

Fig. 127. C^v_4.

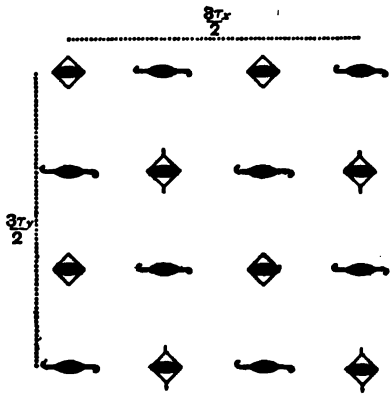

Fig. 128. C^v_4.

§ 3. Tetragonal Hemihedry of the Second Sort.

The group D_{2d} is derived from D_2 by multiplying by a dexion in a plane parallel to one axis (w) and bisecting the gle between the other two; therefore groups D^m_{2d} are derived om D_2^m (or Q^m) by combining with an operation $S(t)$ in plane parallel to one set of axes and bisecting the angle etween the other two sets which brings the system of axes o self-coincidence. In order that this last condition may e fulfilled the plane of $S(t)$ must ntly be parallel to ne of the planes $NRFA$, $DELM$ o 6 *; we shall take t parallel to $NRFA$ †.

Again, since the axis w is a f the second sort for D_{2d}, therefore D^m_{2d} ntains somorphous with rotatory-reflexions ; th s 4-al axes of the second sort. The 2 axes of the group Q^m from d groups D^m_{2d} can be derived no rotation-axes parall f the 4-al axes of the ach case. Suppose

other groups Q^m we may the arrangement of all

f) in a plane parallel to to $NRFA$.

that we have a group containing the screw A_{π,t_1} about an axis a, and the operation $S(t)$ in a plane σ making an angle $\frac{\pi}{4}$ with a. Then $A_{\pi,t_1}.S(t) =$ the translation t_1, followed by $A_{\pi,0}.S$ and then by the translation $t =$ the translation t_1, followed by a rotatory-reflexion of angle $\frac{\pi}{2}$ about an axis perpendicular to a, lying in σ, and passing through the intersection of a and σ (p. 68), and then by the translation t. This is equivalent to a rotatory-reflexion of angle $\frac{\pi}{2}$ about an axis easily found (p. 154).

Of course the translation-groups of Q^m must be specialized, Γ_o and Γ_o' to Γ_t, and Γ_o'' and Γ_o''' to $\Gamma_{t'}$.

In groups D_{2d}^m, derived from Q^1, Q^3, and Q^7, τ_x and τ_y are parallel to AL and AM of Fig. 126; in groups derived from Q^6, Q^8, and Q^9, τ_x and τ_y are parallel to ML and AF. This will be clear from a consideration of pp. 137 and 138.

The arrangement of all the axes and planes of any group D_{2d}^m is given when we know the subgroups Q^m and C_{2v}^m.

Again, two groups derived from the same group Q^m and having the same subgroup C_{2v}^m are identical. For if $\{Q^m, S(t)\}$ and $\{Q^m, S'(t')\}$ have the same subgroup C_{2v}^m, C_{2v}^m must contain both $S(t)$ and $S'(t')$. Therefore, since $\{Q^m, S(t)\}$ has C_{2v}^m as a subgroup and C_{2v}^m contains $S'(t')$, $\{Q^m, S(t)\}$ contains $S'(t')$.

Hence $\{Q^m, S(t)\}$ and $\{Q^m, S'(t')\}$ are identical (p. 165).

It follows from the translations in every case that if the plane $NRFA$ is a gliding-plane of any group D_{2d}^m, so are the parallel planes through A_1' and A_1''; and if a plane through L parallel to $NRFA$ is a gliding-plane, so is the parallel plane through M. Hence we need only consider operations $S(t)$ in the plane $NRFA$ (σ_d) and in the parallel plane through M (σ_{d_1}). The corresponding operations will be denoted by $S_d(t)$ and $S_{d_1}(t)$ respectively.

§ 4. First suppose the translation-group to be Γ_t.

We have

$$D_{2d}^1 = \{Q^1, S_d\} = \{Q^1, S_d(\tau_x+\tau_y)\}^*.$$

Subgroup C_{2v}^{11}; 4-al axes A, A''' †.

* $\{Q^1, S_d\}$ and $\{Q^1, S_d(\tau_x+\tau_y)\}$ are identical because they have the same subgroup C_{2v}^{11}; the reasoning is the same in similar cases.

† These are the points where the axis and a plane of a 4-al axis of the second sort meet.

$$D^2_{2d} = \{Q^1, S_d(\tau_z)\} = \{Q^1, S_d(\tau_x+\tau_y+\tau_z)\}.$$

Subgroup C^{13}_{2v}, 4-al axes N, N_3.

$$D^3_{2d} = \{Q^3, S_d\} = \{Q^3, S_d(\tau_x+\tau_y)\}.$$

Subgroup C^{11}_{2v}; 4-al axes A_1', A_1''.

$$D^4_{2d} = \{Q^3, S_d(\tau_z)\} = \{Q^3, S_d(\tau_x+\tau_y+\tau_z)\}.$$

Subgroup C^{13}_{2v}; 4-al axes N_1, N_2.

No operation in the plane σ_{d_1} brings the system of axes of Q^1 or Q^3 to self-coincidence.

Similarly, we have from Q^6

$$D^5_{2d} = \{Q^6, S_d\}.$$

Subgroup C^1_{2v}; 4-al axes A, A_1', A_1'', A'''.

$$D^6_{2d} = \{Q^6, S_d(\tau_z)\}.$$

Subgroup C^3_{2v}; 4-al axes N, N_1, N_2, N_3.

$$D^7_{2d} = \left\{Q^6, S_{d_1}\left(\frac{\tau_x+\tau_y}{2}\right)\right\}.$$

Subgroup C^6_{2v}; 4-al axes F.

$$D^8_{2d} = \left\{Q^6, S_{d_1}\left(\frac{\tau_x+\tau_y}{2}+\tau_z\right)\right\}.$$

Subgroup C^{10}_{2v}; 4-al axes R.

The operations S_{d_1}, $S_{d_1}(\tau_z)$, $S_d\left(\frac{\tau_x+\tau_y}{2}\right)$, $S_d\left(\frac{\tau_x+\tau_y}{2}+\tau_z\right)$ do not bring the system of axes of Q^3 to self-coincidence.

§ 5. Now suppose the translation-group to be Γ_t'.

From Q^7 we have

$$D^9_{2d} = \{Q^7, S_d\}.$$

Subgroup C^{20}_{2v}; 4-al axes A, A_1.

$$D^{10}_{2d} = \{Q^7, S_d(\tau_z)\}.$$

Subgroup C^{21}_{2v}; 4-al axes A_1''', A'''.

Operations in the plane σ_{d_1} and

$$S_d\left(\frac{\tau_x+\tau_y}{2}\right), \quad S_d\left(\frac{\tau_x+\tau_y}{2}+\tau_z\right)$$

do not bring the system of axes of Q^7 to self-coincidence.

From Q^8 and Q^9 we must obtain groups D_{2d}^m, whose subgroups C_{2v}^m are derived from a group C_2^3, whose axes have the arrangement of Fig. 90 (p. 177), **e** and **h** being rotation-axes; that is, whose subgroups are C_{2v}^{18} or C_{2v}^{19}. Hence (p. 200), we have only

$$D_{2d}^{11} = \{Q^8, S_d\}.$$

Subgroup C_{2v}^{18}; 4-al axes A, A_1', A_1'', A'''.

$$D_{2d}^{12} = \left\{Q^9, S_d\left(\frac{\tau_x+\tau_y+\tau_z}{2}\right)\right\}.$$

Subgroup C_{2v}^{19}; 4-al axes D, middle point of N_1N_3.

§ 6. TETRAGONAL TETARTOHEDRY.

These groups have rotations and screws of angles $\frac{\pi}{2}$ about a series of parallel axes; for the point-group C_4 has a 4-al rotation-axis.

Let e be such an axis and $A_{\frac{\pi}{4},t}$ the reduced screw about it; then t is either $0, \frac{\tau_z}{2}, \tau_z$, or $\frac{3\tau_z}{2}$. Each group C_4^m has a subgroup C_2^m.

First let the translation-group be Γ_t. Let e meet a principal plane (the plane of the paper) in E (Fig. 129); let $EE_1 = 2\tau_x$, $EE_2 = 2\tau_y$, $EE_3 = 2\tau_x + 2\tau_y$; and let G_1, G_2, F be the middle points of EE_1, EE_2, EE_3 respectively.

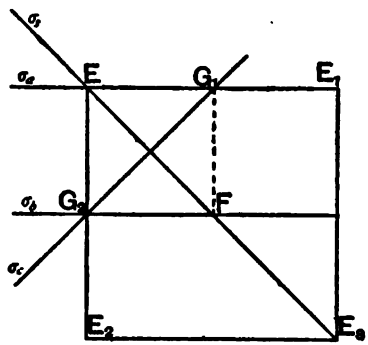

Fig. 129.

Then axes similar to e pass through E_1, E_2, and E_3.

Now, since $A_{\frac{\pi}{2},t}$ about e, followed by the translation EE_1, is equivalent to an operation $B_{\frac{\pi}{2},t}$ about an axis f, parallel to e and passing through F, therefore f is also an axis of the group similar to e. Again, since $A_{\frac{\pi}{2},t}.B_{\frac{\pi}{2},t}$ is by Euler's construction equivalent to an operation $C_{\pi,2t}$ about an axis g_1, parallel to e and passing through G_1; therefore g_1 (and similarly g_2,

parallel to e and passing through G_2) is a screw-axis of the group of angle π and translation $2t$.

There can be no axis not belonging to one of the series $\mathfrak{e}, \mathfrak{f}, \mathfrak{g}_1, \mathfrak{g}_2$, obtained by transforming e, f, g_1, g_2 respectively by the translations of the group. Putting $t = 0, \frac{\tau_z}{2}, \tau_z, \frac{3\tau_z}{2}$ in turn, we have

$C_4^1 = \{A_{\frac{\pi}{2}, 0}, \Gamma_t\}$. Subgroup C_2^1.

$C_4^2 = \{A_{\frac{\pi}{2}, \frac{\tau_z}{2}}, \Gamma_t\}$. Subgroup C_2^2.

$C_4^3 = \{A_{\frac{\pi}{2}, \tau_z}, \Gamma_t\}$. Subgroup C_2^1.

$C_4^4 = \{A_{\frac{\pi}{2}, \frac{3\tau_z}{2}}, \Gamma_t\}$. Subgroup C_2^2.

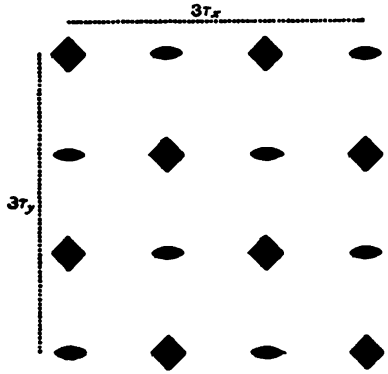

Fig. 180. C_4^1.

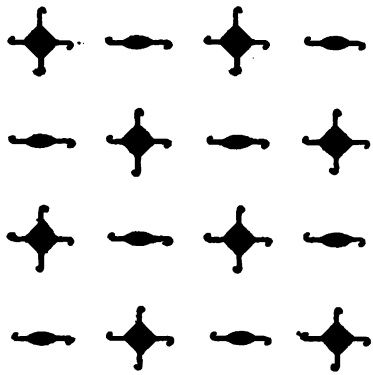

Fig. 181. C_4^2.

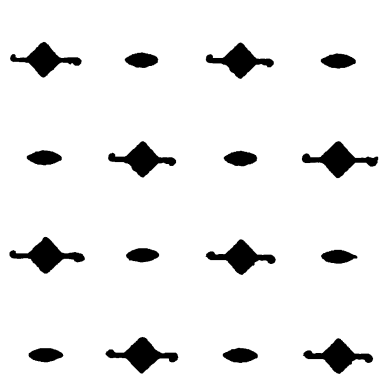

Fig. 182. C_4^3.

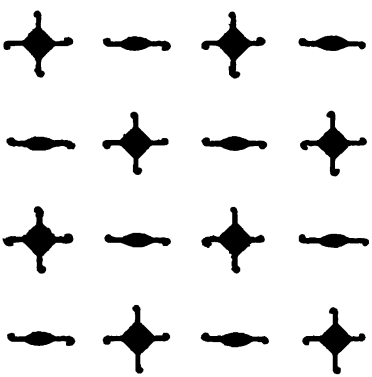

Fig. 183. C_4^4.

Note that e and f may be regarded as 2-al rotation-axes if $t = 0$ or τ_z, and as screw-axes of angle π and translation τ_z if $t = \frac{\tau_z}{2}$ or $\frac{3\tau_z}{2}$.

The arrangement of the axes of these groups is shown in Figs. 130 to 133; ◆ denotes a 4-al rotation-axis, and , , , screw-axes of angle $\frac{\pi}{2}$ and translations $\frac{\tau_z}{2}$, τ_z, $\frac{3\tau_z}{2}$ respectively. The only difference between C_2^2 and C_4^4 is that the screws of one are left-handed and of the other are right-handed.

§ 7. Now let the translation-group be Γ_t'

(primitive triplet, $\tau_y+\tau_z, \tau_z+\tau_x, \tau_x+\tau_y$).

Take now $EE_1 = \tau_x$, $EE_2 = \tau_y$ (*not* $2\tau_x$, $2\tau_y$) in Fig. 129. Then if the axis e of an operation $A_{\frac{\pi}{2}, t}$ passes through E an axis e_3 similar to e passes through E_3, for e_3 is the result of transforming e by $\tau_x+\tau_y$. The group has an operation $B_{\frac{\pi}{2}, t+\tau_z}$ about f, for this is equivalent to $A_{\frac{\pi}{2}, t}$ followed by $\tau_x+\tau_z$. Similarly, it has operations $C_{\pi, 2t+\tau_z}$ about g_1 and g_2.

Hence the axes of the group are a series (**e**) of screw-axes of angle $\frac{\pi}{2}$ and translation t, a series (**f**) of screw-axes of angle $\frac{\pi}{2}$ and translation $t+\tau_z$, and two series ($\mathbf{g}_1$ and $\mathbf{g}_2$) of axes of angle π and translation $2t+\tau_z$ derived from e, f, g_1, and g_2 by transforming by the translations.

A group C_4^m whose translation-group is Γ_t', and which contains screws of angle $\frac{\pi}{2}$ and translation t, contains also screws of angle $\frac{\pi}{2}$ and translation $t+\tau_z$; hence $\{A_{\frac{\pi}{2}, t}, \Gamma_t'\}$ and $\{A_{\frac{\pi}{2}, t+\tau_z}, \Gamma_t'\}$ are identical.

We have therefore only the two groups

$$C_4^5 = \{A_{\frac{\pi}{2}, 0}, \Gamma_t'\}. \qquad \text{Subgroup } C_2^3.$$

$$C_4^6 = \{A_{\frac{\pi}{2}, \frac{\tau_z}{2}}, \Gamma_t'\}. \qquad \text{Subgroup } C_2^3.$$

The arrangement of the axes is shown in Figs. 134, 135.

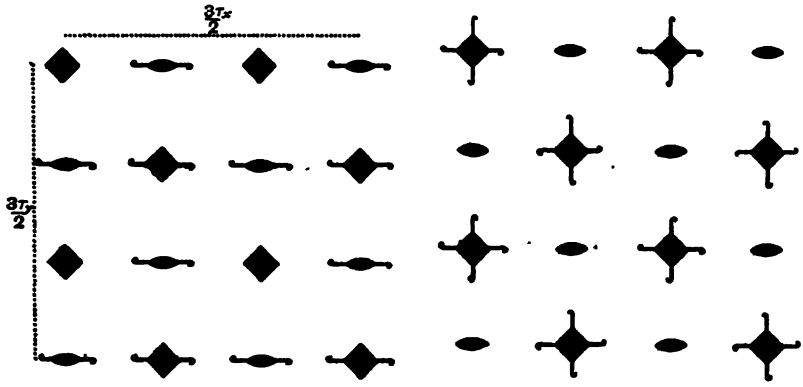

Fig. 134. C_4^5. Fig. 135. C_4^6.

§ 8. Tetragonal Paramorphy.

Since C_{4h} is derived from C_4 by multiplying by an inversion, all groups C_{4h}^m are derived from those groups C_4^m which satisfy the condition of p. 165—that is, from C_4^1, C_4^3, C_4^5, C_4^6—by multiplying by an inversion which brings the system of axes to self-coincidence.

To fulfil this last condition the inversion must take place about a point which lies in an axis or halfway between two neighbouring 4-al axes*. We denote the inversion in these two cases by I and I_1 respectively.

If the group C_{4h}^m has a centre of symmetry in one axis it has centres of symmetry in every axis.

We have then

$C_{4h}^1 = \{C_4^1, I\}$. Subgroup C_{2h}^1.

$C_{4h}^2 = \{C_4^3, I\}$. Subgroup C_{2h}^1.

$C_{4h}^3 = \{C_4^1, I_1\}$. Subgroup C_{2h}^4.

$C_{4h}^4 = \{C_4^3, I_1\}$. Subgroup C_{2h}^4.

$C_{4h}^5 = \{C_4^5, I\}$. Subgroup C_{2h}^3.

$C_{4h}^6 = \{C_4^6, I_1\}$. Subgroup C_{2h}^3.

* That is, at the middle point of *EF*, Fig. 129.

There are no more groups, for I_1 does not bring the system of axes of C_4^5 to self-coincidence, nor I that of C_4^6. The arrangement of the symmetry- and gliding-planes is given by the consideration of the subgroups C_{2h}^m.

§ 9. Tetragonal Hemimorphy.

Since C_{4v} is derived from C_4 by multiplying by a reflexion in a plane parallel to the 4-al axis, therefore groups C_{4v}^m are derived from those groups C_4^m which satisfy the condition of p. 165—that is, from C_4^1, C_4^3, C_4^5, C_4^6—by multiplying by an operation $S(t)$ whose plane σ is parallel to the axes and which brings the system of axes to self-coincidence.

Since C_{4v} has four subgroups C_s, whose planes make angles of $\frac{\pi}{4}$ with one another, C_{4v}^m has four subgroups C_s^m, whose planes intersect at the same angle. Not all these subgroups can be such that their planes are parallel to two translations of some primitive triplet and perpendicular to the third; therefore at least one of these subgroups is either C_s^3 or C_s^4, and each group C_{4v}^m has a symmetry-plane or a gliding-plane whose translation is τ_z.

We may take then $t = 0$ or τ_z; and in that case, since $S(t)$ brings the system of axes of C_4^m to self-coincidence, σ joins two neighbouring axes of the series **e** and **f**, two neighbouring axes of the series **e** and $\mathbf{g}_1$*, two neighbouring axes of the series **f** and $\mathbf{g}_2$, or two neighbouring axes of the series $\mathbf{g}_1$ and $\mathbf{g}_2$. We shall denote σ in these cases by σ_s, σ_a, σ_b, and σ_c respectively (Fig. 129), and the corresponding operations by $S_s(t)$, $S_a(t)$, $S_b(t)$, and $S_c(t)$ respectively ($t = 0$ or τ_z). σ_a and σ_b are only geometrically distinguishable in the case of C_4^5 and C_4^6.

First suppose the translation-group is Γ_t.

We have then

$$C_{4v}^1 = \{C_4^1, S_s\} = \{C_4^1, S_a\} \dagger.$$

$$C_{4v}^2 = \{C_4^1, S_c\}.$$

$$C_{4v}^3 = \{C_4^3, S_s\} = \{C_4^3, S_a(\tau_z)\}.$$

$$C_{4v}^4 = \{C_4^3, S_c\}.$$

* The case in which σ joins two neighbouring axes of the series **e** and $\mathbf{g}_2$ is not geometrically distinguishable from this.

† The group $\{C_4^1, S_s\}$ has σ_a as a symmetry-plane; therefore by p. 165 $\{C_4^1, S_s\}$ and $\{C_4^1, S_a\}$ are identical.

$$C_{4v}^5 = \{C_4^1, S_s(\tau_z)\} = \{C_4^1, S_a(\tau_z)\}.$$

$$C_{4v}^6 = \{C_4^1, S_c(\tau_z)\}.$$

$$C_{4v}^7 = \{C_4^3, S_s(\tau_z)\} = \{C_4^3, S_a\}.$$

$$C_{4v}^8 = \{C_4^3, S_c(\tau_z)\}.$$

The arrangement of the axes and planes of these groups is shown in Figs. 136 to 143.

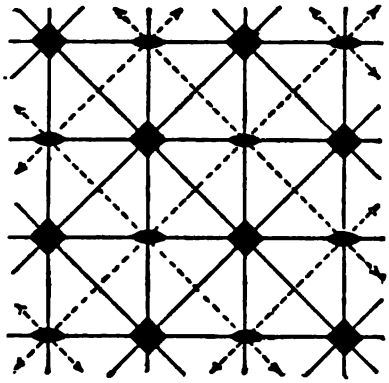

Fig. 136. C_{4v}^1.

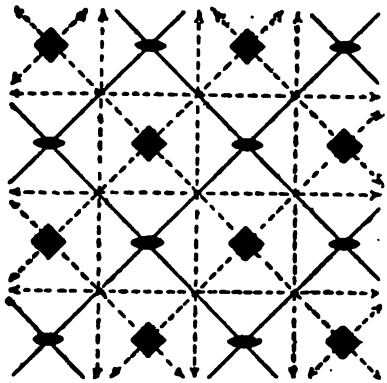

Fig. 137. C_{4v}^2.

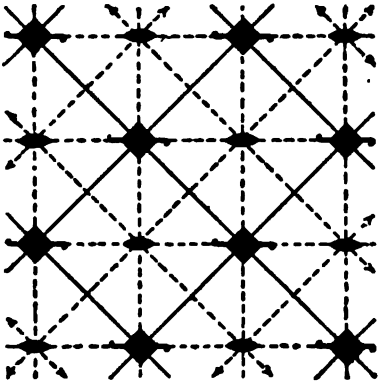

Fig. 138. C_{4v}^3.

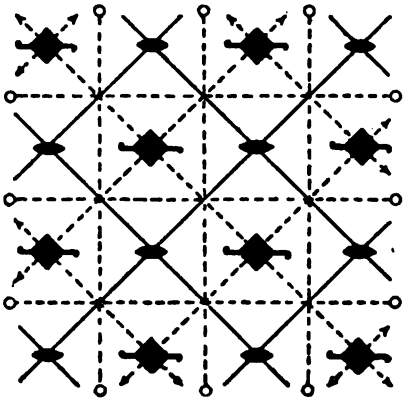

Fig. 139. C_{4v}^4.

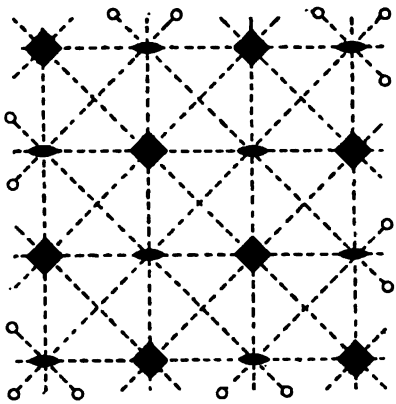

Fig. 140. C_{4v}^{5}.

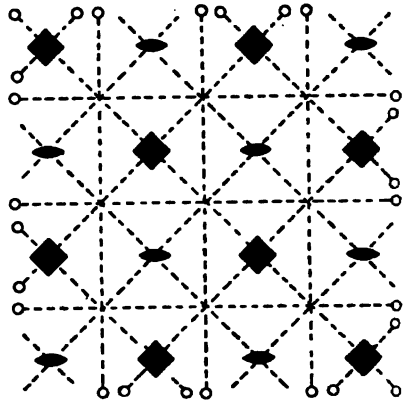

Fig. 141. C_{4v}^{6}.

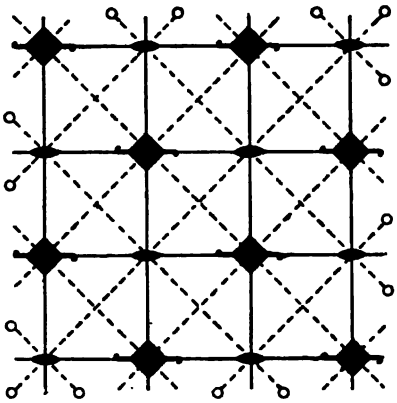

Fig. 142. C_{4v}^{7}.

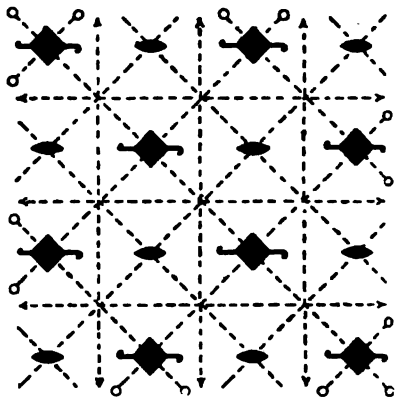

Fig. 143. C_{4v}^{8}.

§ 10. If the translation-group is Γ_t' we have the following groups (Figs. 144 to 147) *.

$$C_{4v}^{9} = \{C_4^{\,5}, S_a\} = \{C_4^{\,5}, S_a\} = \{C_4^{\,5}, S_b(\tau_z)\}.$$

$$C_{4v}^{10} = \{C_4^{\,5}, S_a(\tau_z)\} = \{C_4^{\,5}, S_b\} = \{C_4^{\,5}, S_a(\tau_z)\}.$$

$$C_{4v}^{11} = \{C_4^{\,6}, S_c\}.$$

$$C_{4v}^{12} = \{C_4^{\,6}, S_c(\tau_z)\}.$$

* In these figures a broken line means a plane of translation τ_z, a broken line with dark circles means a plane of translation $\frac{1}{2}(\tau_x+\tau_z)$ or $\frac{1}{2}(\tau_y+\tau_z)$, and one with dark circles with lines through them means a plane of translation $\frac{1}{2}(\tau_x-\tau_z)$ or $\frac{1}{2}(\tau_y-\tau_z)$.

There are no more groups C_{4v}^m, for S_c, $S_c(\tau_z)$ do not bring the system of axes of C_4^5 to self-coincidence, nor S_a, $S_a(\tau_z)$, S_s, $S_s(\tau_z)$ the system of C_4^6 *.

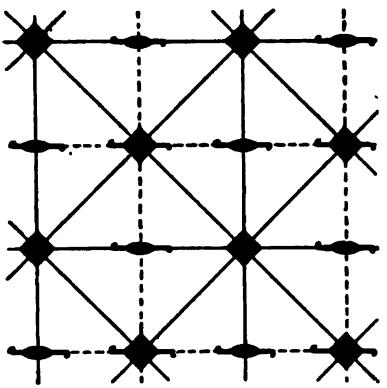

Fig. 144. C_{4v}^9.

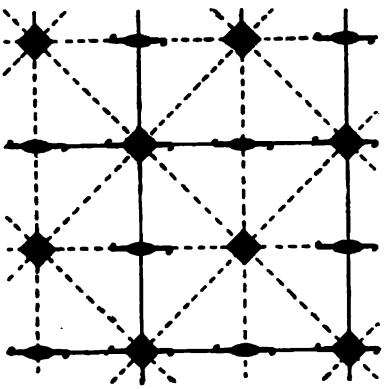

Fig. 145. C_{4v}^{10}.

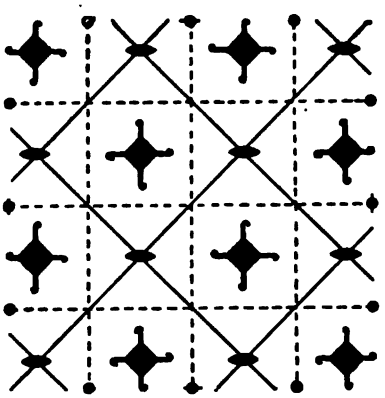

Fig. 146. C_{4v}^{11}.

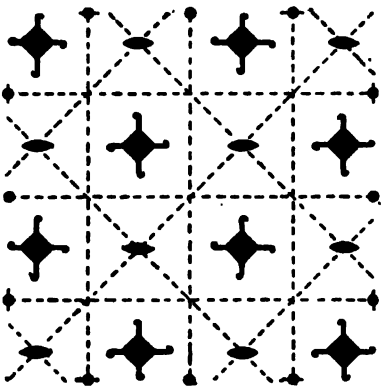

Fig. 147. C_{4v}^{12}.

§ 11. Tetragonal Enantiomorphy.

Since the point-group D_4 is derived from C_4 by multiplying by a rotation about a 2-al axis perpendicular to the axis of C_4, therefore the groups D_4^m are derived from C_4^m by multiplying

* It will be remembered that an operation of the second sort must bring right-handed screw-axes into coincidence with left-handed and vice versa (p. 156).

by a rotation or screw of angle π about an axis perpendicular to the axes of C_4^m which brings the system formed by these axes to self-coincidence. Because D_4 has four subgroups C_2, therefore D_4^m has four subgroups C_2^m; by reasoning similar to that of § 9 it may be shown that one of these subgroups at least is C_2^3. Hence every group D_4^m has a *rotation* of angle π. The axis of such a rotation must evidently have a position similar either to EF, EG_1, FG_2, or G_1G_2 of Fig. 129 (p. 202). We shall denote these lines by u_s, u_a, u_b, u_c respectively, and rotations through π about them by U_s, U_a, U_b, U_c.

However, the existence of the translation $2\tau_y$ in the case of Γ_t and of $\tau_y+\tau_z$ in the case of Γ_t' shows that $\{C_4^m, U_a\}$ and $\{C_4^m, U_b\}$ are always identical.

§ 12. Let the translation-group be Γ_t.
We have then

$$D_4^1 = \{C_4^1, U_s\} = \{C_4^1, U_a\};$$
$$D_4^2 = \{C_4^1, U_c\}.$$

The arrangement of the axes of these groups in each member of a set of principal planes of interval τ_z is shown in Figs. 148, 149; 2-al rotation-axes are represented by thick lines, screw-axes by thick broken lines.

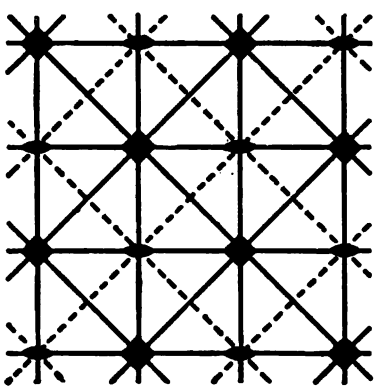

Fig. 148. D_4^1.

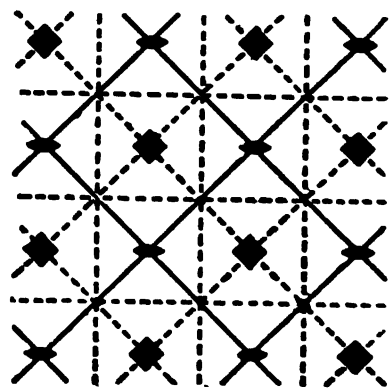

Fig. 149. D_4^2.

A similar arrangement holds for

$$D_4^3 = \{C_4^3, U_s\} = \{C_4^2, U_a\},$$
$$D_4^4 = \{C_4^2, U_c\},$$

except that the axes of the subgroups C_2^m do not intersect. Axes parallel to EG_1 lie in each member of a set of principal planes of interval τ_z; axes parallel to EF, EG_2, G_1G_2 lie in the planes obtained by transforming this set by $\frac{\tau_z}{4}$, $\frac{\tau_z}{2}$, $\frac{3\tau_z}{4}$ respectively.

The same holds for

$$D_4^7 = \{C_4^4, U_g\} = \{C_4^4, U_a\},$$
$$D_4^8 = \{C_4^4, U_c\},$$

if we substitute $-\tau_z$ for τ_z.

For
$$D_4^5 = \{C_4^3, U_g\} = \{C_4^3, U_a\},$$
$$D_4^6 = \{C_4^3, U_c\}$$

axes parallel to EG_1 and EG_2 lie in each member of a set of principal planes of interval τ_z; axes parallel to EF and G_1G_2 lie in the planes halfway between any two consecutive members of this set.

§ 13. U_c does not bring the system of axes of C_4^5 and C_4^6 whose translation-group is Γ_t' into self-coincidence.

We have then only

$$D_4^9 = \{C_4^5, U_g\} = \{C_4^5, U_a\},$$
$$D_4^{10} = \{C_4^6, U_g\} = \{C_4^6, U_a\},$$

D_4^9 has 2-al axes as in Fig. 150 in each of a set of principal planes of interval τ_z; it has also axes as in Fig. 151 in the planes halfway between any two consecutive members of this set.

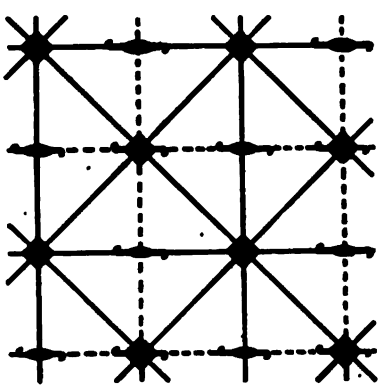

Fig. 150. D_4^9 (1).

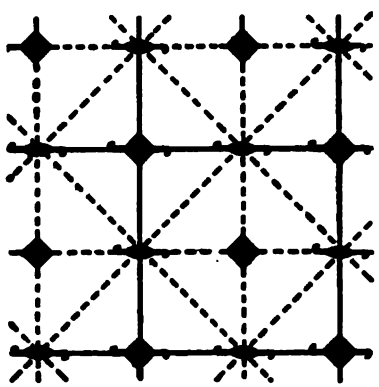

Fig. 151. D_4^9 (2).

D_4^{10} has axes as shown in Fig. 152 in each member of a set of principal planes of interval τ_z; and as shown in Fig. 153 in the planes obtained by transforming the set by $\frac{\tau_z}{4}$. The arrangement of the axes in the planes obtained by transforming the set by $\frac{\tau_z}{2}$ and $\frac{3\tau_z}{4}$ is given by turning Figs. 152 and 153 respectively through a right angle.

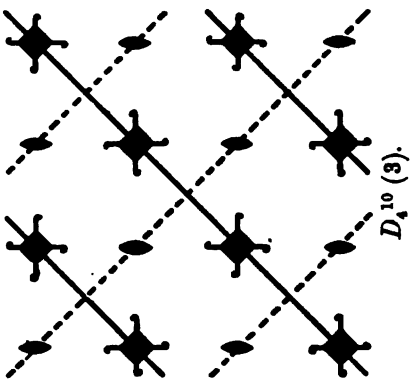

Fig. 152. D_4^{10} (1).

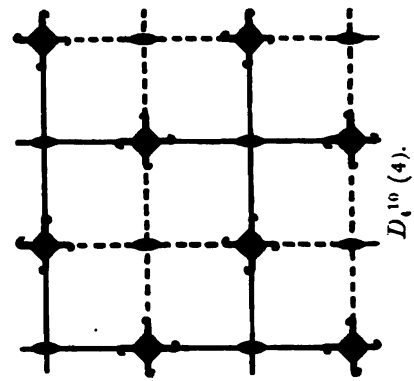

Fig. 153. D_4^{10} (2).

§ 14. TETRAGONAL HOLOHEDRY.

D_{4h} is derived from D_4 by multiplying by an inversion: therefore the groups D_{4h}^m are derived from those groups D_4^m which satisfy the condition of p. 165—that is, D_4^1, D_4^2, D_4^5, D_4^6, D_4^9, D_4^{10}—by multiplying by an inversion (about a point P) which brings the system of axes to self-coincidence.

This last condition is evidently only fulfilled if the inversion brings the system of axes of the subgroup C_4^m and of each of the subgroups C_2^m to self-coincidence; conversely it *is* fulfilled if the inversion brings the system of axes of C_4^m and of *one* of the subgroups C_2^m—say that one whose axes are parallel to $EF(u_3)$ of Fig. 129—into self-coincidence (p. 166). It is sufficient to investigate the cases in which P is on the axis e or halfway between e and f; for each group C_{4h}^m has a centre of symmetry in one or other of these positions, and each group D_{4h}^m has a subgroup C_{4h}^m.

§ 15. First, let the translation-group be Γ_t. In groups derived from D_4^1, D_4^2, D_4^5, and D_4^6 P may evidently be at the point of intersection of e and an axis parallel to u_3, halfway between two such points of intersection, on an axis parallel to

u_s halfway between e and f, or halfway between e and f and halfway between axes parallel to u_s; for in such positions and in such only is it a centre of symmetry for the system of axes of C_4^m *and* the system of that subgroup C_2^m whose axes are parallel to u_s. The inversions in the four cases are denoted by I, I', I_1, and I_1' respectively.

We have therefore:—

$D_{4h}^1 = \{D_4^1, I\}$. Subgroups C_{4h}^1; C_{4v}^1.

$D_{4h}^2 = \{D_4^1, I'\}$. Subgroups C_{4h}^1; C_{4v}^5.

$D_{4h}^3 = \{D_4^1, I_1\}$. Subgroups C_{4h}^3; C_{4v}^2.

$D_{4h}^4 = \{D_4^1, I_1'\}$. Subgroups C_{4h}^3; C_{4v}^6.

$D_{4h}^5 = \{D_4^2, I\}$. Subgroups C_{4h}^1; C_{4v}^2.

$D_{4h}^6 = \{D_4^2, I'\}$. Subgroups C_{4h}^1; C_{4v}^6.

$D_{4h}^7 = \{D_4^2, I_1\}$. Subgroups C_{4h}^3; C_{4v}^1.

$D_{4h}^8 = \{D_4^2, I_1'\}$. Subgroups C_{4h}^3; C_{4v}^5.

$D_{4h}^{10} = \{D_4^5, I\}$ *. Subgroups C_{4h}^2; C_{4v}^3.

$D_{4h}^9 = \{D_4^5, I'\}$. Subgroups C_{4h}^2; C_{4v}^7.

$D_{4h}^{12} = \{D_4^5, I_1\}$. Subgroups C_{4h}^4; C_{4v}^4.

$D_{4h}^{11} = \{D_4^5, I_1'\}$. Subgroups C_{4h}^4; C_{4v}^8.

$D_{4h}^{14} = \{D_4^6, I\}$. Subgroups C_{4h}^2; C_{4v}^4.

$D_{4h}^{13} = \{D_4^6, I'\}$. Subgroups C_{4h}^2; C_{4v}^8.

$D_{4h}^{16} = \{D_4^6, I_1\}$. Subgroups C_{4h}^4; C_{4v}^3.

$D_{4h}^{15} = \{D_4^6, I_1'\}$. Subgroups C_{4h}^4; C_{4v}^7.

§ 16. Now let the translation-group be Γ_t'. In groups derived from D_4^2, P must be either at the point of intersection of e with an axis parallel to u_s, or halfway between two such points of intersection. P cannot lie halfway between e and f, for there is no group C_{4h}^m derived from C_4^5 with a centre of

* The natural notation has been altered to bring it into accordance with that of Schoenflies. Reference to his "Krystallsysteme und Krystallstructur" will be thereby facilitated.

symmetry in that position. Similarly in groups derived from D_4^{10}, P must be either halfway between e and f and on an axis parallel to u_s, or halfway between e and f and halfway between axes parallel to u_s.

$D_{4h}^{17} = \{D_4^9, I\}$. Subgroups C_{4h}^5; C_{4v}^9.

$D_{4h}^{18} = \{D_4^9, I'\}$. Subgroups C_{4h}^5; C_{4v}^{10}.

$D_{4h}^{19} = \{D_4^{10}, I_1\}$. Subgroups C_{4h}^6; C_{4v}^{11}.

$D_{4h}^{20} = \{D_4^{10}, I_1'\}$. Subgroups C_{4h}^6; C_{4v}^{12}.

CHAPTER XXI

RHOMBOHEDRAL GROUPS.

§ 1. The point-groups of the rhombohedral system all possess one 3-al rotation-axis; therefore the isomorphous space-groups have screws of angle $\frac{2\pi}{3}$ about a single series of parallel axes. The planes perpendicular to these axes will be called principal planes. We shall take $2\tau_z$ as the primitive translation parallel to any reduced screw $A_{\frac{2\pi}{3},t}$; so that $t = 0, \frac{2\tau_z}{3}$, or $\frac{4\tau_z}{3}$.

The translation-group of any rhombohedral group is either Γ_h or Γ_{rh} (p. 159). Since Γ_h contains no translation whose component in the direction of τ_z is less than $2\tau_z$, therefore the translations of all screws of a group whose translation-group is Γ_h are the same. On the other hand Γ_{rh} contains a translation whose component parallel to τ_z is $\frac{2\tau_z}{3}$; therefore if a group whose translation-group is Γ_{rh} has screws whose translation is t, it has screws whose translations are $t + \frac{2\tau_z}{3}$ and $t + \frac{4\tau_z}{3}$; that is, it has screws whose translations are $0, \frac{2\tau_z}{3}$, and $\frac{4\tau_z}{3}$, since t has one of the values $0, \frac{2\tau_z}{3}, \frac{4\tau_z}{3}$.

The series of axes obtained by transforming any axis e by the translations will (as before) be denoted by $\mathfrak{e}$.

§ 2. RHOMBOHEDRAL TETARTOHEDRY.

Groups of this class have a single series of parallel 3-al axes; their operations are obtained by multiplying the operations of the translation-group by a single operation of the form $A_{\frac{2\pi}{3},t}$.

First, let Γ_h be the translation-group; all screws of such groups C_3^m have the same translation (p. 161).

Suppose the axis e of $A_{\frac{2\pi}{3},t}$ to be perpendicular to the plane of the paper and to meet it in E. Take $EE_1 = 2\tau_1$, $EE_2 = 2\tau_2$, $EE_3 = -2\tau_1 - 2\tau_2$, $EE_4 = 2\tau_1 + 2\tau_2$, $EE_5 = -2\tau_1$, $EE_6 = -2\tau_2$, using the notation of p. 139 (Fig. 154).

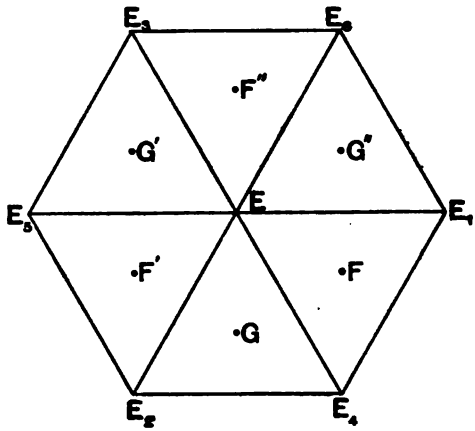

Fig. 154.

Then, since EE_1, EE_2, EE_3, EE_4, EE_5, EE_6 are translations of the group

$$\{A_{\frac{2\pi}{3},t}, \Gamma_h\},$$

therefore axes similar to e pass through the points E_1, E_2, E_3, E_4, E_5, E_6.

Since the operation $A_{\frac{2\pi}{3},t}$, followed by the translation EE_1 is equivalent to a screw $B_{\frac{2\pi}{3},t}$ about an axis f, parallel to e, passing through the centre F of the triangle EE_1E_4 (p. 148), therefore the group has f as a screw-axis; similarly it has a screw $C_{\frac{2\pi}{3},t}$ about an axis g passing through the centre of the triangle EE_2E_4. It has no axes other than those obtained by transforming e, f, g by the translations of Γ_h; for if it had a screw $H_{\frac{2\pi}{3},t}$ about any other axis the group would have a translation $A_{\frac{2\pi}{3},t} \cdot (H_{\frac{2\pi}{3},t})^{-1}$—not belonging to Γ_h which is impossible.

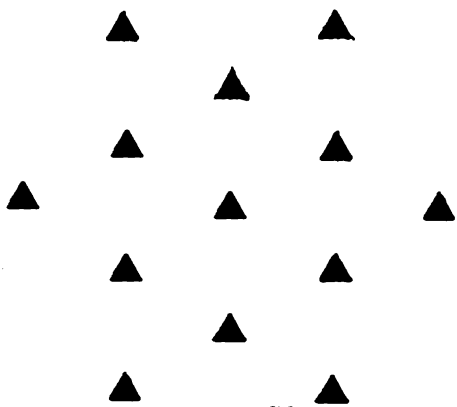

Fig. 155. C_3^1.

We have then the three groups :—

$C_3^1 = \{A_{\frac{2\pi}{3}, 0}, \Gamma_h\}$.

$C_3^2 = \{A_{\frac{2\pi}{3}, \frac{2\tau_z}{3}}, \Gamma_h\}$.

$C_3^3 = \{A_{\frac{2\pi}{3}, \frac{4\tau_z}{3}}, \Gamma_h\}$.

The group C_3^3 contains the operation $A_{\frac{2\pi}{3}, -\frac{2\tau_z}{3}}$; for this is equivalent to

$$A_{\frac{2\pi}{3}, \frac{4\tau_z}{3}},$$

followed by the translation $-2\tau_z$. Therefore the only difference between C_3^2 and C_3^3 is that the screws of one are left-handed and those of the other right-handed.

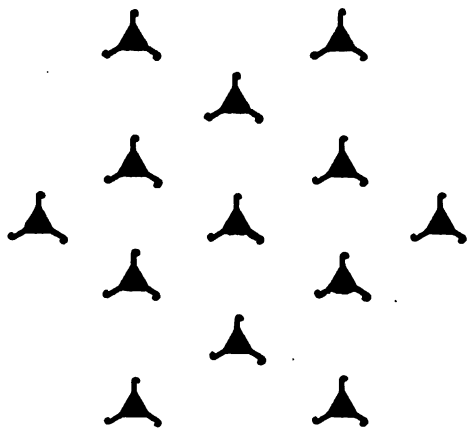

Fig. 156. C_3^2.

The arrangement of the axes of the groups C_3^1, C_3^2, C_3^3 is shown in Figs. 155, 156, 157; a 3-al rotation-axis is represented by ▲, and 3-al screw-axes of translation

$$\frac{2\tau_z}{3}, \frac{4\tau_z}{3}$$

by ▲, ▲, respectively.

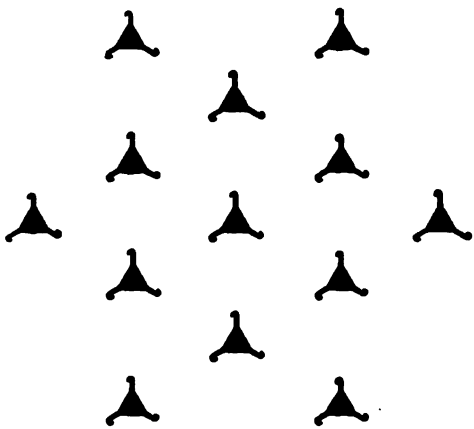

Fig. 157. C_3^3.

§ 3. Now suppose the translation-group to be Γ_h. Take any point O in the axis e of a screw $A_{\frac{2\pi}{3}, t}$, and let OE_1, OE_2, OE_3 represent the translations $2\tau'$, $2\tau''$, $2\tau'''$ of p. 141.

Let the plane $E_1E_2E_3$ cut e in E (Figs. 158, 159).

Then axes exactly similar to e pass through E_1, E_2, E_3. Now the screw $A_{\frac{2\pi}{3}, t}$, followed by the translation OE_1 (whose component parallel to e is $\frac{2\tau_z}{3}$), is equivalent to a screw $B_{\frac{2\pi}{3}, t+\frac{2\tau_z}{3}}$ about an axis f, passing through the point F a third of the way from E_1 to E_2; the group

$$\{A_{\frac{2\pi}{3}, t}, \Gamma_{rh}\}$$

has therefore f, as a screw-axis, whose translation is $t + \frac{2\tau_z}{3}$; similarly it has a screw-axis g, whose translation is $t + \frac{4\tau_z}{3}$, passing through the point G halfway between F and E_2. As before it may be seen that the group has no axes except those obtained by transforming e, f, g by the translations. It is evident that the axes e, f, g have exactly similar positions with regard to all the elements of the group, so that it is indifferent whether we take

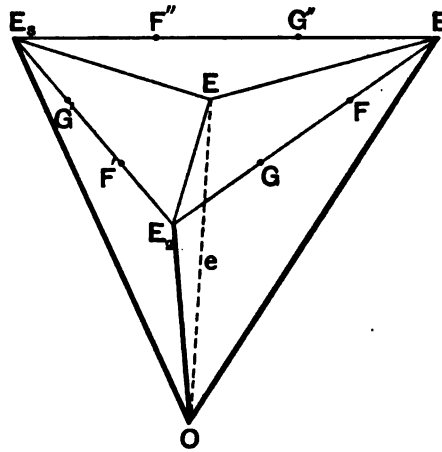

Fig. 158.

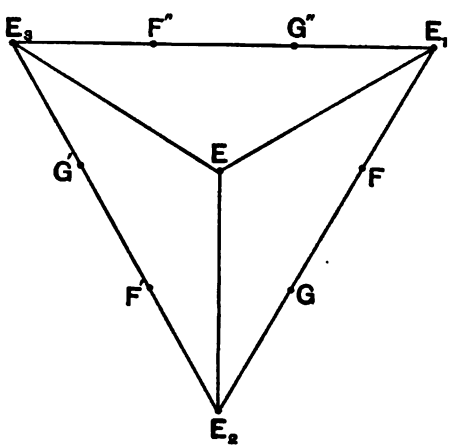

Fig. 159.

$t = 0$ (and therefore $t + \frac{2\tau_z}{3} = \frac{2\tau_z}{3}$, $t + \frac{4\tau_z}{3} = \frac{4\tau_z}{3}$)

or $t = \frac{2\tau_z}{3}$ (and therefore $t + \frac{2\tau_z}{3} = \frac{4\tau_z}{3}$, $t + \frac{4\tau_z}{3} - 2\tau_z = 0$)

or $t = \frac{4\tau_z}{3}$ (and therefore $t + \frac{2\tau_z}{3} - 2\tau_z = 0$, $t + \frac{4\tau_z}{3} - 2\tau_z = \frac{2\tau_z}{3}$).

Hence we have only one group of this sort (Fig. 160).

$$C_3^4 = \{A_{\frac{2\pi}{3},0}, \Gamma_{rh}\} = \{A_{\frac{2\pi}{3},\frac{2\tau_z}{3}}, \Gamma_{rh}\} = \{A_{\frac{2\pi}{3},\frac{4\tau_z}{3}}, \Gamma_{rh}\}.$$

The only difference between the series of axes **f** and **g** is that one series is left-handed and the other right-handed.

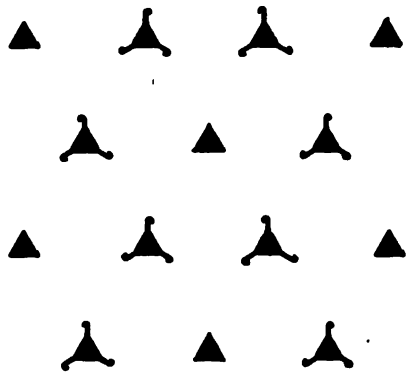

Fig. 160. C_3^4.

§ 4. Hexagonal Tetartohedry of the Second Sort.

Since the group C_{3i} is obtained by multiplying the operations of C_3 by an inversion, therefore all groups C_{3i}^m are obtained by multiplying the operations of those groups C_3^m which satisfy the condition of p. 165—that is, C_3^1 and C_3^4—by an inversion which brings the system of axes of C_3^m into self-coincidence. In order that this last condition may be fulfilled the inversion must take place about a point in an axis or halfway between two axes; if, however, a group C_3^m has a centre of symmetry lying in an axis, it has one lying halfway between two axes and vice versa. For if E (Fig. 154) is a centre of symmetry the point midway between F and G must also be a centre of symmetry, since EE_4 is a translation; and a similar proof holds in the case of C_3^4. It is therefore sufficient to combine C_3^1 and C_3^4 with an inversion about a point in an axis.

All axes of C_3^1 are similar, and therefore we have only one group derivable from C_3^1.

$$C_{3i}^1 = \{C_3^1, I\}.$$

The axes of C_6^3 are not similar, but inversion about a point in the axis f or g does not bring the system of axes to self-coincidence. Hence the only group derivable from C_3^4 is obtained by combining its operations with inversion about a point in a *rotation-axis*.

$$C_{3i}^2 = \{C_3^4, I\}.$$

§ 5. Rhombohedral Hemimorphy.

Exactly as in § 9 of the previous chapter it may be shown that the groups C_{3v} are derived from C_3^1 or C_3^4 by multiplying by an operation S or $S(\tau_z)$ in a plane σ parallel to the axes, which brings the system of axes to self-coincidence.

This last condition is evidently only satisfied if σ passes through a line such as EF or EE_3 of Fig. 154 or a line such as EE_3 of Fig. 159. Planes parallel to the axes through EF and EE_3 will be denoted by σ_s and σ_a respectively.

We have then the following groups :—

$C_{3v}^1 = \{C_3^1, S_s\}$.	Subgroups C_s^3. (3) *.
$C_{3v}^2 = \{C_3^1, S_a\}$.	Subgroups C_s^3. (3).
$C_{3v}^3 = \{C_3^1, S_s(\tau_z)\}$.	Subgroups C_s^4. (3).
$C_{3v}^4 = \{C_3^1, S_a(\tau_z)\}$.	Subgroups C_s^4. (3).
$C_{3v}^5 = \{C_3^4, S_a\}$.	Subgroups C_s^3. (3).
$C_{3v}^6 = \{C_3^4, S_a(\tau_z)\}$.	Subgroups C_s^4. (3).

See Figs. 161 to 166.

§ 6. Rhombohedral Enantiomorphy.

By reasoning exactly similar to that of § 11 of the previous chapter, it may be shown that all groups D_3^m are obtained by multiplying C_3^m by a *rotation* of angle π, which brings the system of axes of C_3^m to self-coincidence. To fulfil this last condition it is necessary that the axis of this rotation should be such a line as EF or EE_3 of Fig. 154 or EF of Fig. 159. We denote rotations of angle π about EF and EE_3 by U_s and U_a respectively.

Since no two members of a primitive pair in a principal

* That is, C_{3v}^1 has three subgroups C_s^m, each of which is C_s^s.

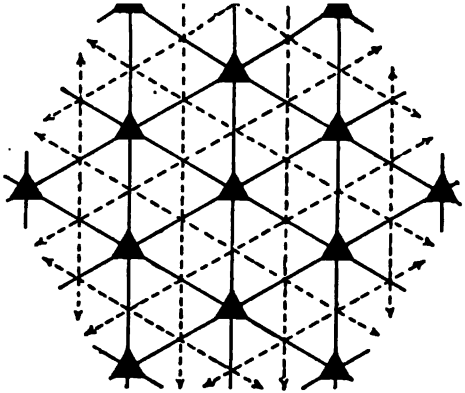

Fig. 161. C_{3v}^{1}.

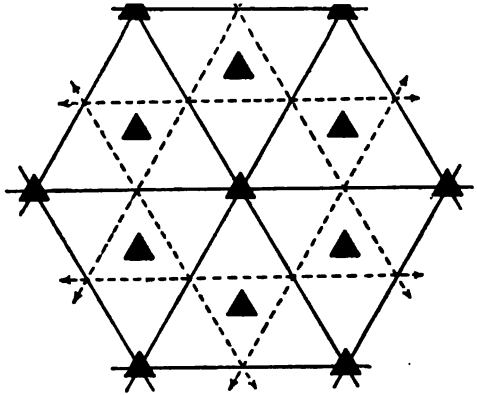

Fig. 162. C_{3v}^{2}.

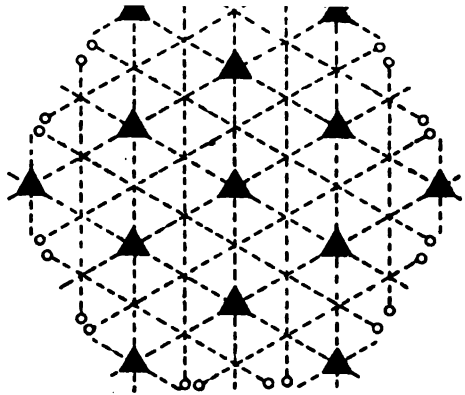

Fig. 163. C_{3v}^{3}.

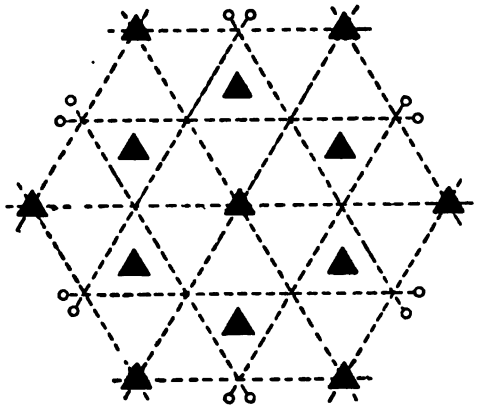

Fig. 164. C_{3v}^{4}.

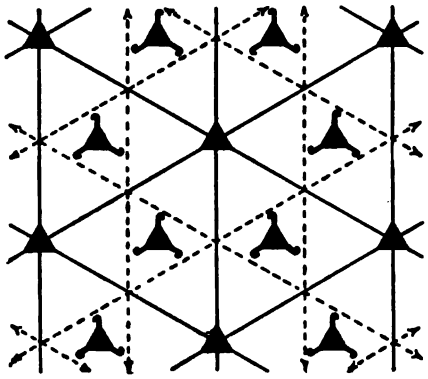

Fig. 165. C_{3v}^{5}.

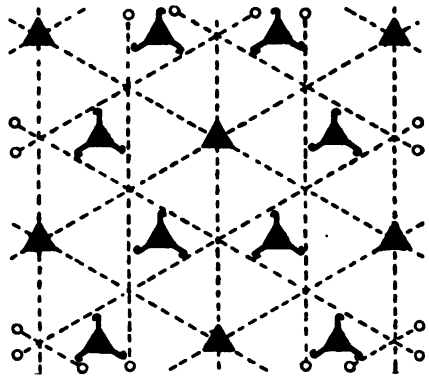

Fig. 166. C_{3v}^{6}.

plane are at right angles each of the three subgroups C_2^m is C_2^3. We have then

$$D_3^1 = \{C_3^1, U_s\}.$$
$$D_3^2 = \{C_3^1, U_a\}.$$

These groups have axes arranged as in Figs. 167, 168, respectively, in each member of a set of principal planes of interval τ_z.

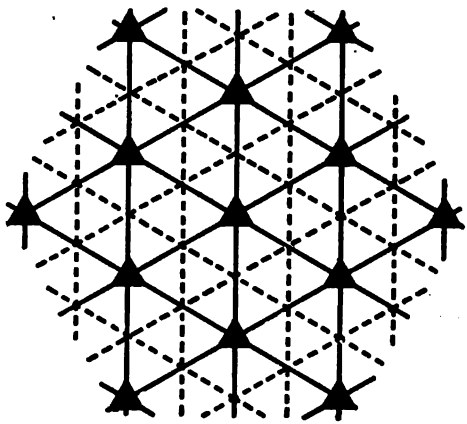

Fig. 167. D_3^1.

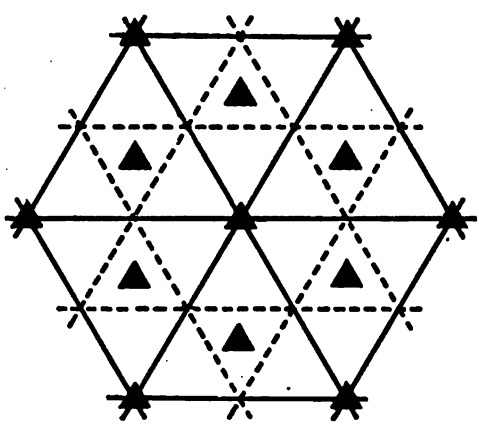

Fig. 168. D_3^2.

The arrangement of the axes of

$$D_3^3 = \{C_3^2, U_s\},$$
$$D_3^4 = \{C_3^2, U_a\}$$

is similar, except that the axes of the subgroups C_2^3 do not intersect. Axes parallel to EF or EE_4 (Fig. 154) lie in each member of a set of principal planes of interval τ_z; axes parallel to EF' or EE_2 lie in planes obtained by transforming this set by $\frac{\tau_z}{3}$; axes parallel to EF'' or EE_1 lie in planes obtained by transforming the set by $\frac{2\tau_z}{3}$.

To

$$D_3^5 = \{C_3^3, U_s\},$$
$$D_3^6 = \{C_3^3, U_a\}$$

the above remarks apply if we put $-\tau_z$ for τ_z.

The translation-group of the above groups is Γ_h; the only group whose translation-group is Γ_{rh} is

$$D_3^7 = \{C_3^4, U_s\}.$$

It has axes as shown in Fig. 169 in each member of a set of principal planes of interval τ_z; and as shown in Figs. 170, 171 in the planes obtained by transforming the set by $\frac{\tau_z}{3}$, $\frac{2\tau_z}{3}$, respectively.

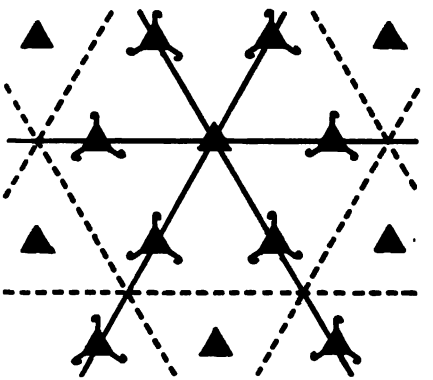

Fig. 169. D_3^7 (1).

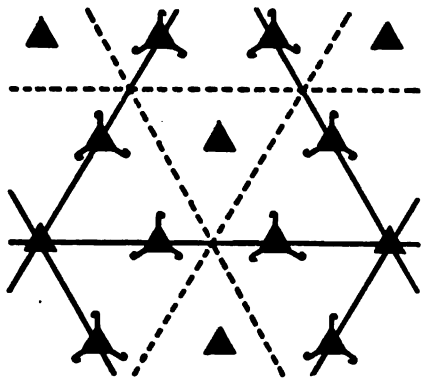

Fig. 170. D_3^7 (2).

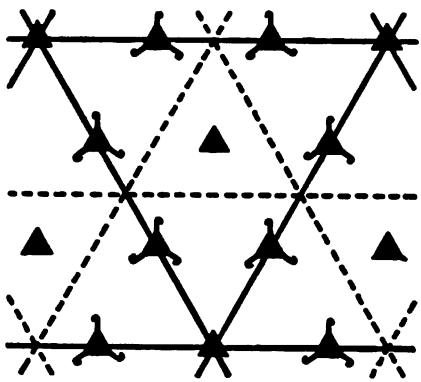

Fig. 171. D_3^7 (3).

§ 7. Rhombohedral Holohedry.

By reasoning very similar to that of § 14 of the previous chapter, it may be shown that all groups D_{3d}^m are obtained by multiplying those groups D_3^m which satisfy the condition of p. 165—that is D_3^1, D_3^2, and D_3^7—by an inversion about a point lying in a 3-al rotation-axis, either at its intersection

with a perpendicular 2-al axis or halfway between two such intersections.

Denoting the inversions in these two cases by I and I', respectively, we have,

$D^1_{3d} = \{D_3^1, I\}$. Subgroups C^1_{3i}; C^2_{3v}; C^3_{2h} (3).

$D^2_{3d} = \{D_3^1, I'\}$. Subgroups C^1_{3i}; C^4_{3v}; C^6_{2h} (3).

$D^3_{3d} = \{D_3^2, I\}$. Subgroups C^1_{3i}; C^1_{3v}; C^3_{2h} (3).

$D^4_{3d} = \{D_3^2, I'\}$. Subgroups C^1_{3i}; C^3_{3v}; C^5_{2h} (3).

$D^5_{3d} = \{D_3^7, I\}$. Subgroups C^2_{3i}; C^5_{3v}; C^3_{2h} (3).

$D^6_{3d} = \{D_3^7, I'\}$. Subgroups C^2_{3i}; C^6_{3v}; C^6_{2h} (3).

The arrangements of the elements of symmetry are given by consideration of the subgroups.

CHAPTER XXII

HEXAGONAL GROUPS.

§ 1. Groups of this system have a series of parallel axes isomorphous with a 6-al (in the case of C_{3h}^m and D_{3h}^m with a 3-al) rotation-axis, these are called principal axes; planes perpendicular to them are (as before) called principal planes. There is only one possible translation-group, namely, Γ_h; for Γ_{rh} has not the symmetry of any point-group of the hexagonal system.

§ 2. Trigonal Paramorphy.

Since C_{3h} is derived from C_3 by multiplying by a reflexion in a plane perpendicular to the 3-al axis, therefore groups C_{3h}^m are derived from those groups C_3^m which satisfy the condition of p. 165, and have Γ_h as their translation-group—that is, from C_3^1 only—by multiplying by an operation $S(t)$ in a principal plane which brings the system of axes of C_3^1 into self-coincidence. To satisfy this last condition t (which must be half a translation in a principal plane) must be 0; for no two axes of C_3^1 are at a distance from one another equal to half a primitive translation. Hence we have only a single group of this class

$$C_{3h}^1 = \{C_3^1, S_h\}. \qquad \text{Subgroup } C_s^1.$$

It has as its symmetry-planes a set of principal planes of interval τ_s.

§ 3. Trigonal Holohedry.

Exactly in the same way it is seen that groups D_{3h}^m can only be derived by multiplying D_3^1 and D_3^2 by a reflexion in a principal plane σ. In this case, however, the position of σ is no longer arbitrary; it must either contain 2-al axes when it is denoted by σ_h, or lie halfway between them when it is denoted by σ_m.

We have then

$D^1_{3h} = \{D_3^1, S_h\}$. Subgroups C^1_{3h}; C^1_{3v}.
$D^2_{3h} = \{D_3^1, S_m\}$. Subgroups C^1_{3h}; C^3_{3v}.
$D^3_{3h} = \{D_3^2, S_h\}$. Subgroups C^1_{3h}; C^2_{3v}.
$D^4_{3h} = \{D_3^2, S_m\}$. Subgroups C^1_{3h}; C^4_{3v}.

§ 4. Hexagonal Tetartohedry.

Groups of this class must contain a series of parallel screw-axes of angles $\frac{\pi}{3}$, whose translations have one of the values $\frac{m\tau_z}{3}(m = 0, 1, 2, 3, 4, 5)$. The translations of all reduced screws of the same angle must be equal, for Γ_h has no translation whose component parallel to the axis is $< 2\tau_z$ (p. 161).

Using the notation of Fig. 154, let the axis e of a screw $A_{\frac{\pi}{3}, t}$ pass through E; then, transforming e by the translations, we see that the groups of this class have a series of similar screw-axes passing through the points $E, E_1, E_2, E_3, \ldots$. (This is consistent with the construction of Fig. 80, p. 148).

Again, they have a series of screw-axes of angle $\frac{2\pi}{3}$ and translation $2t$ through points such as F and G, since $A_{\frac{\pi}{3}, t}$ about e, followed by a similar screw about e_4, is equivalent, by Euler's construction, to a screw $B_{\frac{2\pi}{3}, 2t}$ about f. They have also a series of screw-axes of angle π and translation $3t$ lying halfway between any two screw-axes similar to e, for $A_{\frac{\pi}{3}, t}$ about e, followed by $B_{\frac{2\pi}{3}, 2t}$ about f, is equivalent to a screw $D_{\pi, 3t}$ about an axis h halfway between E and E_1.

There are evidently six groups of this class

$$C_6^1 = \{A_{\frac{\pi}{3}, 0}, \Gamma_h\}.$$

$$C_6^2 = \{A_{\frac{\pi}{3}, \frac{\tau_z}{3}}, \Gamma_h\}.$$

$$C_6^3 = \{A_{\frac{\pi}{3}, \frac{5\tau_z}{3}}, \Gamma_h\}.$$

$$C_6^4 = \{A_{\frac{\pi}{3}, \frac{2\tau_z}{3}}, \Gamma_h\}.$$

$$C_6^5 = \{A_{\frac{\pi}{3}, \frac{4\tau_z}{3}}, \Gamma_h\}.$$

$$C_6^6 = \{A_{\frac{\pi}{3}, \tau_z}, \Gamma_h\}.$$

The arrangements of the axes of C_6^1, C_6^2, C_6^4, and C_6^6 are shown in Figs. 172 to 175 *; the arrangements of C_6^3 and C_6^5 only differ from those of C_6^2 and C_6^4 respectively, in having right-handed screws instead of left-handed and vice versa.

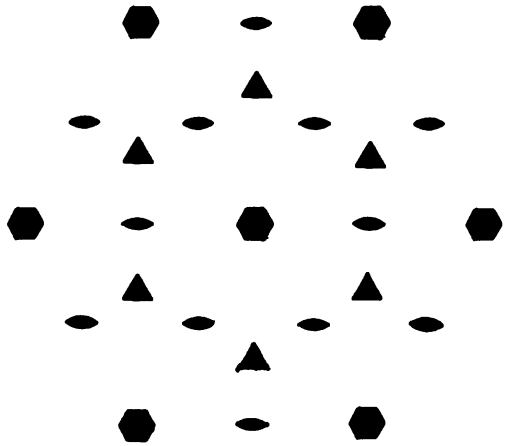

Fig. 172. C_6^1.

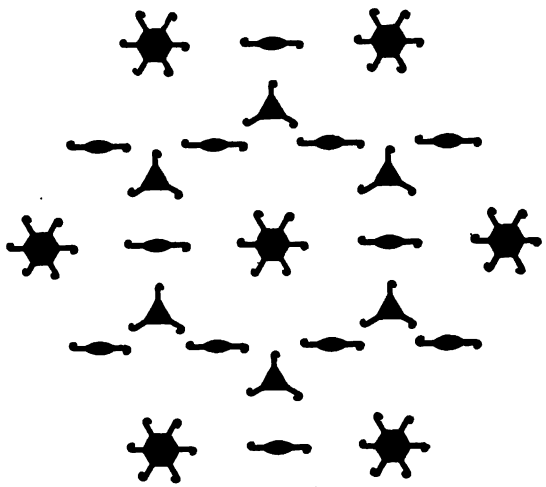

Fig. 173. C_6^2.

* ⬢, , , , represent screw-axes of angle $\frac{\pi}{3}$ and translations 0, $\frac{\tau_3}{3}$, $\frac{2\tau_3}{3}$, τ_3 respectively.

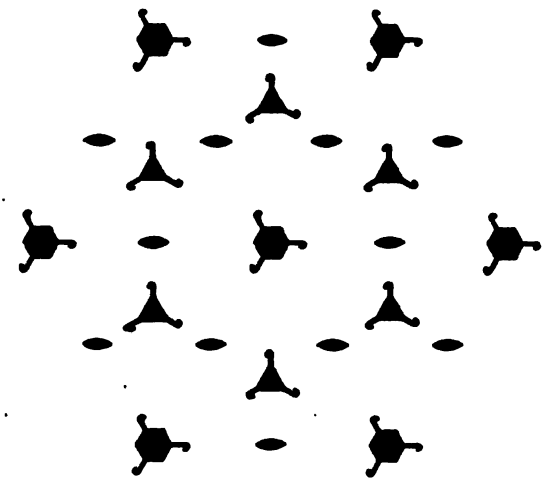

Fig. 174. C_6^4.

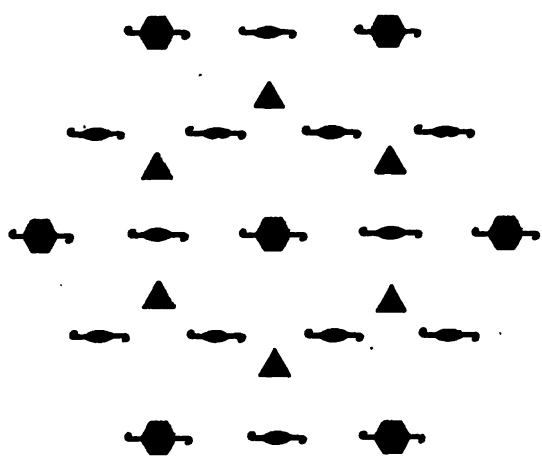

Fig. 175. C_6^6.

§ 5. Hexagonal Paramorphy.

Since C_{6h} is derived from C_6 by multiplying by an inversion, therefore the groups C_{6h}^m are derived from those groups C_6^m which satisfy the condition of p. 165—that is, C_6^1 and C_6^5—by multiplying by an inversion (about a point P) which brings the system of axes to self-coincidence. To fulfil this condition P must be such a point as E or the middle of EE_4 (Fig. 154). Since, however, an inversion about the middle point of EE_4

followed by the translation E_4E is equivalent to an inversion about E, it is sufficient to take P at E.

We have then

$$C^1_{6h} = \{C_6^1, I\}. \qquad \text{Subgroup } C_s^1.$$
$$C^2_{6h} = \{C_6^6, I\}. \qquad \text{Subgroup } C_s^1.$$

Both groups have for symmetry-planes a set of principal planes of interval τ_z. The centres of symmetry lie in these planes in the case of C^1_{6h}, and halfway between these planes in the case of C^2_{6h}.

§ 6. Hexagonal Hemimorphy.

Exactly as in § 9 of chapter xx it may be shown that all groups C^m_{6v} are obtained by combining C_6^1 or C_6^6 with an operation S or $S(\tau_z)$ in a plane σ parallel to the axes which brings the system of axes to self-coincidence. If this last condition is fulfilled σ must pass through such a line as EE_4 or EF of Fig. 154 (cf. § 5 of chap. xxi).

We have then (Figs. 176 to 179):—

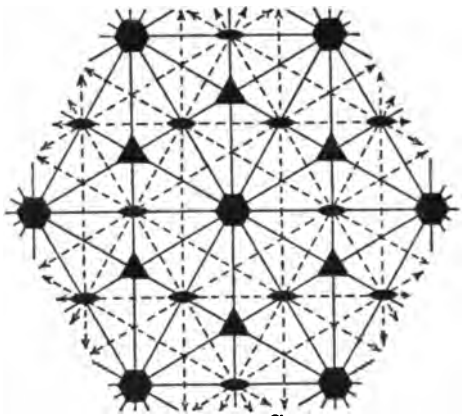

Fig. 176. C^1_{6v}.

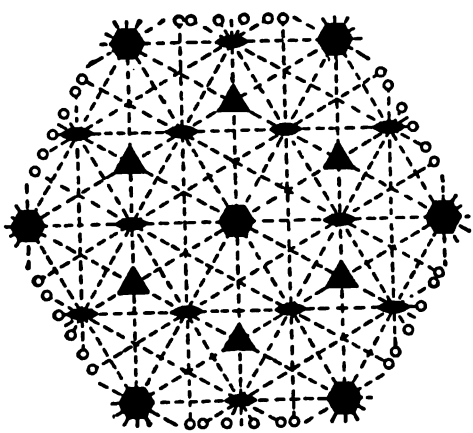

Fig. 177. C^2_{6v}.

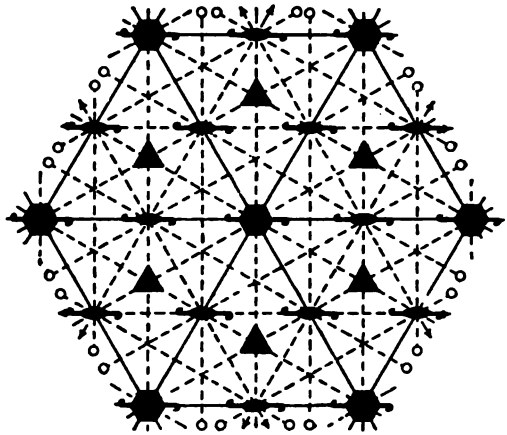

Fig. 178. C_{6v}^3.

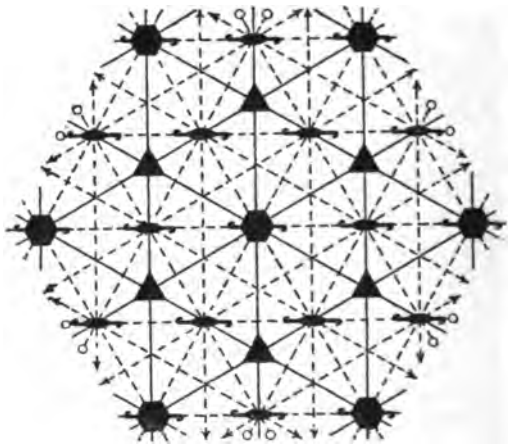

Fig. 179. C_{6v}^4.

$C_{6v}^1 = \{C_6^1, S_a\} = \{C_6^1, S_s\}$.
Subgroups $C_s^3(6)$.

$C_{6v}^2 = \{C_6^1, S_a(\tau_z)\} = \{C_6^1, S_s(\tau_z)\}$.
Subgroups $C_s^4(6)$.

$C_{6v}^3 = \{C_6^6, S_a\} = \{C_6^6, S_s(\tau_z)\}$.
Subgroups $C_s^3(3)$; $C_s^4(3)$.

$C_{6v}^4 = \{C_6^6, S_a(\tau_z)\} = \{C_6^6, S_s\}$.
Subgroups $C_s^4(3)$; $C_s^3(3)$.

§ 7. HEXAGONAL ENANTIOMORPHY.

Exactly as in the case of D_3^m (p. 220) it may be shown that the groups D_6^m are derived from C_6^m by multiplying by a rotation about a line such as EE_4 or EF.

Since, however, the screw of angle $\frac{\pi}{3}$ about e, followed by a rotation through π about EE_4, is equivalent to a rotation through π about a line parallel to EF, the groups $\{C_6^m, U_s\}$ and $\{C_6^m, U_a\}$ are identical.

We have then

$$D_6^1 = \{C_6^1, U_a\}.$$
$$D_6^2 = \{C_6^2, U_a\}.$$
$$D_6^3 = \{C_6^3, U_a\}.$$
$$D_6^4 = \{C_6^4, U_a\}.$$
$$D_6^5 = \{C_6^5, U_a\}.$$
$$D_6^6 = \{C_6^6, U_a\}.$$

D_6^1 has axes arranged as in Fig. 180 in each member of a set of principal planes of interval τ_z.

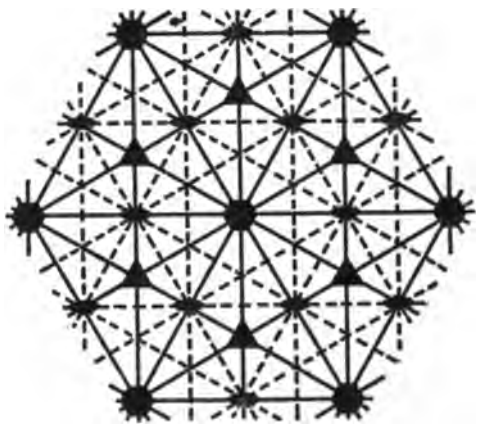

Fig. 180. D_6^1.

For the other groups the arrangement is similar, but the axes of the various subgroups C_2^m do not all intersect.

In the case of D_6^2 axes parallel to EE_4 lie in each member of a set of principal planes of interval τ_z, while axes parallel to EG, EE_2, EF', EE_5, EG' lie in the planes obtained by transforming this set by $\frac{\tau_z}{6}$, $\frac{\tau_z}{3}$, $\frac{\tau_z}{2}$, $\frac{2\tau_z}{3}$, $\frac{5\tau_z}{6}$ respectively.

The same holds for D_6^3 if we put $-\tau_z$ for τ_z.

In the case of D_6^4 axes parallel to EE_4 and EF' lie in each member of a set of principal planes of interval τ_z, while axes parallel to EG and EE_5 lie in the planes obtained by trans-

forming this set by $\frac{\tau_z}{3}$, and axes parallel to EE_2 and EG' lie in the planes obtained by transforming the set by $\frac{2\tau_z}{3}$. The same holds for D_6^5 if we put $-\tau_z$ for τ_z.

In the case of D_6^6 axes parallel to EE_4, EE_2, and EE_5 lie in each member of a set of principal planes of interval τ_z, and axes parallel to EG, EF', and EG' lie in the planes halfway between any two consecutive members of this set.

§ 8. HEXAGONAL HOLOHEDRY.

Groups D_{6h}^m are derived from D_6^1 and D_6^6, just as groups C_{6h}^m were derived from C_6^1 and C_6^6, by multiplying by an inversion about a point P, which may be taken as lying in a 6-al axis. If P lies at a point where a 6-al axis of D_6^1 is met by a perpendicular rotation-axis or halfway between two points where it is so met, inversion about P evidently brings the system of axes of that group to self-coincidence, and otherwise not. Again if P lies at a point where a 6-al screw-axis of D_6^6 is met by a perpendicular rotation-axis parallel to EF, or by a perpendicular rotation-axis parallel to EE_4, inversion about P evidently brings the system of axes of that group to self-coincidence and otherwise not.

Hence we have

$D_{6h}^1 = \{D_6^1, I\}$. Subgroups C_{6h}^1; C_{6v}^1.

$D_{6h}^2 = \{D_6^1, I'\}$. Subgroups C_{6h}^1; C_{6v}^2.

$D_{6h}^3 = \{D_6^6, I_a\}$. Subgroups C_{6h}^2; C_{6v}^3.

$D_{6h}^4 = \{D_6^6, I_\beta\}$. Subgroups C_{6h}^2; C_{6v}^4.

CHAPTER XXIII

REGULAR GROUPS.

§ 1. All groups of the regular system have one of the three translation-groups Γ_r, Γ_r', Γ_r'', whose primitive triplets may be taken as $2\tau_x, 2\tau_y, 2\tau_z$; $\tau_y+\tau_z, \tau_z+\tau_x, \tau_x+\tau_y$;

and $\tau_y+\tau_z-\tau_x, \tau_z+\tau_x-\tau_y, \tau_x+\tau_y-\tau_z$; respectively.

In each case τ_x, τ_y, τ_z are equal in magnitude.

§ 2. Regular Tetartohedry.

Since the point-group T has a subgroup Q and four subgroups C_3, whose axes make equal angles with the axes of Q, therefore any group T^m has a subgroup Q^m and four subgroups C_3^m, whose axes make equal angles with the axes of Q^m.

The subgroup Q^m is specialized (by the fact that its orthorhombic translation-group is specialized to a regular translation-group) in such a way that the parallelepipedon p of Figs. 117 to 125 (p. 192) is a cube. The axes of the subgroups C_3^m run parallel to the diagonals of this cube; and it is at once evident from the position of these diagonals with regard to the translations, that these subgroups can only be C_3^4, and that hence they all contain rotation-axes. Rotation through $\frac{2\pi}{3}$ about any one of these axes brings the system of axes of Q^m into self-coincidence; hence the arrangement of each of the three series of axes of Q^m must be the same; therefore Q^m must be one of Q^1, Q^4, Q^7, Q^8, or Q^9. The group T^m may be derived from its subgroup Q^m by multiplying by a rotation A through $\frac{2\pi}{3}$ about any one of its 3-al rotation-axes a, for T can be derived from its subgroup Q by multiplying by a similar rotation about one of its 3-al axes.

Since a rotation through $\frac{2\pi}{3}$ about a brings the system of axes to self-coincidence, a must evidently be some diagonal of p. These diagonals have all the same position with regard to the system of axes as a whole (unless the subgroup is Q^7,

when a *must* coincide with AA_1 of Fig. 126). We hence have the same group T^m whichever diagonal we take for a; we shall take AA_1. We have then

$$T^1 = \{Q^1, A\}.$$
$$T^2 = \{Q^7, A\}.$$
$$T^3 = \{Q^8, A\}.$$
$$T^4 = \{Q^4, A\}.$$
$$T^5 = \{Q^9, A\}.$$

The arrangements of the axes of the regular groups are too complicated to be shown readily in diagrams; but the following considerations will make it clear in the present case. The arrangement of the axes of each subgroup C_2^m is the same, and so is that of each subgroup C_3^4; the arrangement of the 3-al axes of any one of the latter is given by Figs. 158 and 159 when the translation-group is known * (Γ_r for T^1 and T^4, Γ_r' for T^2, and Γ_r'' for T^3 and T^5), if the position of one rotation-axis is also known. This is found in the case of T^1, T^2, and T^3 by noticing that if a 2-al and a 3-al rotation-axis intersect at an angle $\cos^{-1}\frac{1}{\sqrt{3}}$, three other 3-al rotation-axes pass through their intersection (cf. p. 60), so that rotation-axes of each subgroup C_3^4 of T^1, T^2, and T^3 pass through the vertex A of p (Fig. 181). In T^4 and T^5 no two 3-al rotation-axes can meet; the relative arrangement is easily found by transforming the axis a by the various screws of angle π, and is shown in Fig. 182.

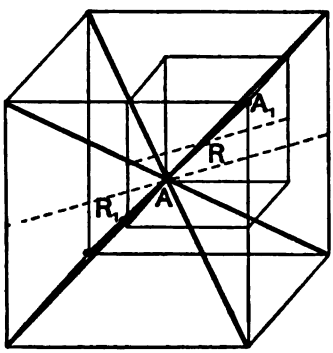

Fig. 181.

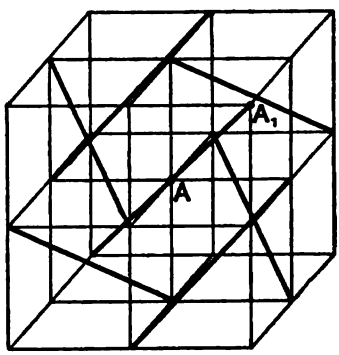

Fig. 182.

* In § 1 the primitive triplets were put in a form the same as that of p. 141; that is, each triplet consists of three equal translations all equally inclined to the 3-al axes, and each equally inclined to the other two.

§ 3. Regular Paramorphy.

Since T_h is derived from T by multiplying by an inversion, the groups T_h^m are derived from T^m by multiplying by an inversion * (about a point P) which brings the system of axes to self-coincidence.

From p. 166 we know that if the inversion brings the system of axes of the subgroup Q^m and of *one* of the subgroups C_3^4 to self-coincidence it brings the whole system of T^m to self-coincidence, and thus forms a group T_h^m from T^m. Now by combining the operations of C_3^4 with an inversion we get C_{3i}^2, and this group has a centre of symmetry in each rotation-axis.

Hence it is sufficient to take P in any given 3-al rotation-axis of T^m; we shall suppose P lies in the diagonal AA_1 of the cube p (Fig. 126), which is always a 3-al rotation-axis of T^m.

If, moreover, the inversion is to bring the system of axes of the subgroup Q^m to self-coincidence, P must be at a vertex or at the centre of p; that is, at A or R of Fig. 126. We see, however, from pp. 195, 197, that the groups obtained from T^3, T^4, and T^5 (whose subgroups are Q^4, Q^8, and Q^9 respectively) by giving P these two positions in turn are not distinct. The arrangement of the axes and planes of T_h^m is given completely when we know the subgroups T^m and Q_h^m.

$T_h^1 = \{T^1, I\}$. Subgroup Q_h^1.

$T_h^2 = \{T^1, I_r\}$. Subgroup Q_h^2.

$T_h^3 = \{T^2, I\}$. Subgroup Q_h^{23}.

$T_h^4 = \{T^2, I_r\}$. Subgroup Q_h^{24}.

$T_h^5 = \{T^3, I\}$. Subgroup Q_h^{25}.

$T_h^6 = \{T^4, I\}$. Subgroup Q_h^{15}.

$T_h^7 = \{T^5, I\}$. Subgroup Q_h^{27}.

§ 4. Regular Hemimorphy.

Since T_d is derived from T by multiplying by a reflexion in a plane through a 3-al and a 2-al axis, therefore the groups T_d^m are derived from T^m by multiplying by an operation $S(t)$ in a plane parallel to the axes of *any* two subgroups C_2^m and C_3^4 which brings the system of axes of T^m to self-coincidence. This last condition is fulfilled (p. 166) if $S(t)$ brings the system of the subgroup Q^m and that of one subgroup C_3^4 to

* All groups T^m satisfy the condition of p. 165.

self-coincidence. We may take the plane of $S(t)$ parallel to *NRFA* (Fig. 126).

Every subgroup C_{3v}^m of T_d^m is either C_{3v}^5 or C_{3v}^6; for every subgroup C_3^m of T^m is C_3^4. Therefore every group T_d^m has a gliding-plane passing through a 3-al rotation-axis, whose translation is either zero or half the primitive translation parallel to that axis. Hence, in order to obtain all groups of the form $\{T^m, S(t)\}$, it is sufficient to take *NRFA* as the plane of $S(t)$, and to take $t = 0$ or $=$ half a translation in the direction AA_1.

$S(t)$ forms with Q^m a group D_{2d}^m; since no group of this class can be formed from Q^4 (p. 199), no group T_d^m can be formed from T^4. No group D_{2d}^m can be formed from Q^8 by an operation $S(t)$ in the plane *NRFA* if $t = \frac{1}{2}(\tau_x + \tau_y + \tau_z)$, nor from Q^9 if $t = 0$. Hence we have

$$T_d^1 = \{T^1, S_d\}. \qquad \text{Subgroup } D_{2d}^1.$$

$$T_d^2 = \{T^2, S_d\}. \qquad \text{Subgroup } D_{2d}^2.$$

$$T_d^3 = \{T^3, S_d\}. \qquad \text{Subgroup } D_{2d}^{11}.$$

These three groups have each four subgroups C_{3v}^5.

$$T_d^4 = \{T^1, S_d(\tau_x + \tau_y + \tau_z)\}. \qquad \text{Subgroup } D_{2d}^2.$$

$$T_d^5 = \{T^2, S_d(\tau_x + \tau_y + \tau_z)\}^*. \qquad \text{Subgroup } D_{2d}^{10}.$$

$$T_d^6 = \left\{T^3, S_d\left(\frac{\tau_x + \tau_y + \tau_z}{2}\right)\right\}. \qquad \text{Subgroup } D_{2d}^{12}.$$

These three groups have each four subgroups C_{3v}^6.

§ 5. Regular Enantiomorphy.

Since O may be derived from T by a rotation through π about any axis perpendicular to a 3-al and a 2-al axis, the groups O^m may be derived from T^m by multiplying by an operation $U_{\pi,t}$ about an axis u perpendicular to the axes of any two subgroups C_3^4 and C_2^m which brings the system of axes to self-coincidence. We may take u parallel to *LM* (Fig. 126).

Since U forms with a subgroup C_3^4 of T^m a subgroup D_3^7 of O^m—for D_3^7 is the only group derivable from C_3^4 by an operation such as U—and since D_3^7 has a 2-al rotation-axis meeting each 3-al rotation-axis, therefore it is sufficient to

* This is equivalent to $\{T^2, S_d(\tau_z)\}$ since $\tau_x + \tau_y$ is a translation.

consider the cases in which the translation t of U is equal to zero and in which u intersects AA_1.

Of course two groups are identical if they are derived from the same group T^m by rotations through π about two lines which are both rotation-axes in the same subgroup D_3^7.

In the case of T^1 u must evidently pass through a point such as A or R of Fig. 126 (cf. Fig. 181), for a rotation through π about u brings the system of axes to self-coincidence.

$O^1 = \{T^1, U\}$. Subgroups D_4^1 (3) *.

$O^2 = \{T^1, U_r\}$. Subgroups D_4^5 (3).

Similarly in the case of T^2 we have

$O^3 = \{T^2, U\}$. Subgroups D_4^9 (3).

$O^4 = \{T^2, U_r\}$. Subgroups D_4^{10} (3).

Since T^3 has a translation $\tau_x+\tau_y+\tau_z$ the groups $\{T^3, U\}$ and $\{T^3, U_r\}$ are identical.

$O^5 = \{T^3, U\}$. Subgroups D_4^9 (3).

In the case of T^4 u must evidently pass through the middle point of A_1R or the middle point of RA (see Fig. 183), or through similar points.

$O^6 = \{T^4, U_1\}$.
Subgroups D_4^4 (3).

$O^7 = \{T^4, U_2\}$.
Subgroups D_4^8 (3).

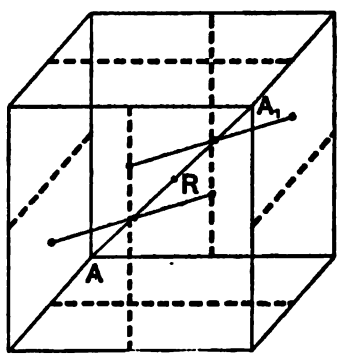

Fig. 183.

These two groups are alike except that where one has left-handed screws the other has right-handed, and vice versa.

For groups derived from T^5 the axis u must have a position similar to that in the case of groups derived from T^4. Since, however, the group T^5 has a translation $\tau_x+\tau_y+\tau_z$, the groups $\{T^5, U_1\}$ and $\{T^5, U_2\}$ are identical.

$O^8 = \{T^5, U\}$. Subgroups D_4^{10} (3).

* That is, the group has three subgroups D_4^m each of which is D_4^1. Each group O^m has also four subgroups D_3^7.

§ 6. Regular Holohedry.

Since O_h is derived from O by multiplying by an inversion, therefore groups O_h^m are derived from those groups O^m which satisfy the condition of p. 165—that is, O^1, O^2, O^3, O^4, O^5, and O^8—by multiplying by an inversion (about a point P) which brings the system of axes to self-coincidence. This last condition is fulfilled if and only if the inversion brings to self-coincidence the system of axes of the subgroup T^m and of that subgroup C_2^m whose axes are parallel to the line u of the previous section (p. 166).

Hence it is necessary and sufficient to consider those cases in which P lies at a vertex or at the centre of the cube p *, and also on an axis such as u, or halfway between two rotation-axes parallel to u.

We have then readily

$O_h^1 = \{O^1, I\}$. Subgroups T_h^1; D_{3d}^5 (4); D_{4h}^1 (3).

$O_h^2 = \{O^1, I_r\}$. Subgroups T_h^2; D_{3d}^6 (4); D_{4h}^4 (3).

$O_h^3 = \{O^2, I\}$. Subgroups T_h^1; D_{3d}^6 (4); D_{4h}^9 (3).

$O_h^4 = \{O^2, I_r\}$. Subgroups T_h^2; D_{3d}^5 (4); D_{4h}^{12} (3).

From O^1 and O^2 we get the same group O_h^m if we multiply by an inversion about A_1 as if we multiply by an inversion about A, for AA_1 is half a translation of O^1 and O^2. This is not the case for O^3; we have in fact

$O_h^5 = \{O^3, I\}$. Subgroups T_h^3; D_{3d}^5 (4); D_{4h}^{17} (3).

$O_h^6 = \{O^3, I_1\}$. Subgroups T_h^3; D_{3d}^6 (4); D_{4h}^{18} (3).

There can be no group $\{O^3, I_r\}$, for the centre of p is not on or halfway between axes parallel to u. For a similar reason there can be no groups $\{O^4, I\}$ or $\{O^4, I_1\}$, but there are two groups derivable from O^4 by taking P either halfway between A and A_1 or at the same distance from A on the line AA_1 but on the opposite side of A (R_1 of Fig. 181).

$O_h^7 = \{O^4, I_r\}$. Subgroups T_h^4; D_{3d}^5 (4); D_{4h}^{19} (3).

$O_h^8 = \{O^4, I_{r_1}\}$. Subgroups T_h^4; D_{3d}^6 (4); D_{4h}^{20} (3).

O^5 and O^8 have a translation $\tau_x+\tau_y+\tau_z$; therefore $\{O^5, I\}$

* With an exception in the case of O_h^8.

and $\{O^5, I_r\}$ on the one hand and $\{O^8, I\}$ and $\{O^8, I_r\}$ on the other are identical.

$O_h^9 = \{O^5, I\}$. Subgroups T_h^5; D_{3d}^5 (4); D_{4h}^{17} (3).

$O_h^{10} = \{O^8, I\}$. Subgroups T_h^7; D_{3d}^5 (4); D_{4h}^{20} (3).

§ 7. The following table shows the distribution of the 230 possible groups of movements which we have now obtained:—

		Γ_{tr}								
	Triclinic System									
C_1	Hemihedry	1							1	2.
C_i	Holohedry	1							1	
	Monoclinic System	Γ_m, Γ_m'								
C_s	Hemihedry	2	2						4	13.
C_2	Hemimorphy	2	1						3	
C_{2h}	Holohedry	4	2						6	
	Orthorhombic System	Γ_o, Γ_o', Γ_o'', Γ_o'''								
C_{2v}	Hemimorphy	10	7	2	3				22	59.
Q	Enantiomorphy	4	2	1	2				9	
Q_h	Holohedry	16	6	2	4				28	
	Tetragonal System	Γ_t, Γ_t'								
C_4'	Tetartohedry of the second sort	1	1						2	68.
D_{2d}	Hemihedry of the second sort	8	4						12	
C_4	Tetartohedry	4	2						6	
C_{4h}	Paramorphy	4	2						6	
C_{4v}	Hemimorphy	8	4						12	
D_4	Enantiomorphy	8	2						10	
D_{4h}	Holohedry	16	4						20	
	Rhombohedral System	Γ_h, Γ_{rh}								
C_3	Tetartohedry	3	1						4	25.
C_{3i}	Hexagonal tetartohedry of the second sort	1	1						2	
C_{3v}	Hemimorphy	4	2						6	
D_3	Enantiomorphy	6	1						7	
D_{3d}	Holohedry	4	2						6	

	Hexagonal System	Γ_h									
C_{3h}	Trigonal paramorphy	1								1	
D_{3h}	Trigonal holohedry	4								4	
C_6	Tetartohedry	6								6	
C_{3h}	Paramorphy	2								2	27.
C_{6v}	Hemimorphy	4								4	
D_6	Enantiomorphy	6								6	
D_{6h}	Holohedry	4								4	

	Regular System	Γ_r	Γ_r'	Γ_r''						
T	Tetartohedry	2	1	2					5	
T_h	Paramorphy	3	2	2					7	
T_d	Hemimorphy	2	2	2					6	36.
O	Enantiomorphy	4	2	2					8	
O_h	Holohedry	4	4	2					10	

CHAPTER XXIV

SPACE-PARTITIONING.

§ 1. We defined (p. 47) the series of points with which any given point is brought to coincidence by the various operations of any group Γ as a 'system of equivalent points,' and proved that in general a point is brought into coincidence with any given equivalent point by one and only one operation of Γ.

§ 2. We will now show how to fill up space with a series of cells each of which contains one point and one only of any equivalent system. Each such cell will be called a *fundamental cell* *.

Take any number of points $P, Q, R, \ldots$ no two of which are equivalent, and draw closed surfaces of very small dimensions round each; now let each surface grow in all directions, the growth in any direction only stopping when further growth in that direction would include, within the surface, a point equivalent to some point already within one or other of the surfaces. When further growth is no longer possible without violating this condition let the portion of space included inside all the closed surfaces be ϕ; let ϕ be brought to $\phi, \phi_1, \phi_2, \phi_3, \ldots$ by all the operations of any group of movements Γ. Any one of these portions is brought to coincidence with any other by some operation of Γ †. We shall show that $\phi, \phi_1, \phi_2, \ldots$ are fundamental cells. We start by proving that $\phi, \phi_1, \phi_2, \ldots$ fill space completely, and that no two of them overlap.

Firstly, they fill space completely; for, if not, suppose there is an interval of space A between ϕ_i and ϕ_m not included in $\phi, \phi_1, \phi_2, \ldots$; let L be the operation bringing ϕ_i to coincide with ϕ, and let L bring ϕ_m to coincide with ϕ_p. Then there is evidently an interval of space A' (with which A is brought to coincide by L) between ϕ and ϕ_p not included in $\phi, \phi_1, \phi_2, \ldots$; and there is no point inside A' equivalent to any point

* German, 'Fundamentalbereich.' The cell must have closed boundaries, but not necessarily a *single* closed boundary.

† Proof as on p. 47.

inside ϕ. Therefore ϕ may be allowed to grow so as to absorb A' without violating the condition that ϕ should contain no two equivalent points; but this would be contrary to the hypothesis that ϕ cannot grow further without violating this condition.

Secondly, no two of $\phi, \phi_1, \phi_2, \ldots$ overlap. For if ϕ_i and ϕ_m overlap so must ϕ and ϕ_p. Let H be a point in the portion of space common to ϕ and ϕ_p; let L' be the operation bringing ϕ_p to coincide with ϕ, and let L' bring H to coincide with K. Then since H is in ϕ_p, K must be in ϕ; but H is in ϕ, and therefore ϕ contains two equivalent points H and K contrary to hypothesis.

No one of the regions $\phi, \phi_1, \phi_2, \ldots$ contains *in its interior* two equivalent points. For if ϕ_i contains two equivalent points P_i, P_i', suppose the operation L of Γ brings ϕ_i into coincidence with ϕ, and P_i, P_i' to coincide with P, P'; then P, P' are two equivalent points lying in ϕ contrary to hypothesis.

Each one of $\phi, \phi_1, \phi_2, \ldots$ contains at least one of any system of equivalent points on its surface or in its interior. For let Q be any point of such a system; then Q must lie in or on one of $\phi, \phi_1, \phi_2, \ldots$, for between them they fill all space; let Q lie in or on ϕ_u. Then, since ϕ_u can be brought to coincidence with *any one* of $\phi, \phi_1, \phi_2, \ldots$ by some operation of Γ, some point equivalent to Q must lie in or on each of $\phi, \phi_1, \phi_2, \ldots$.

The only operation of Γ which can bring any one of the spaces ϕ_v into coincidence with itself is the identical operation. For let L be an operation bringing ϕ_v to self-coincidence, and let L bring any point P inside ϕ_v to P'; then P' is evidently inside ϕ_v, and therefore ϕ_v contains two equivalent points, which is impossible unless P and P' coincide, and hence $L = 1$.

Again ϕ_v cannot be brought to coincide with the same space ϕ_w by two distinct operations L and L' of Γ, for otherwise $L'.L^{-1}$ would be an operation of Γ distinct from unity bringing ϕ_v to self-coincidence.

Now consider any point P *on the boundary* of ϕ_n. Suppose P lies on the boundary between ϕ_n and ϕ_e. Let L be an operation bringing ϕ_e to coincide with ϕ_n, ϕ_n with ϕ_f, and P with P'. Then obviously P' lies on the boundary between ϕ_n and ϕ_f, so that there is at least one point equivalent to P which is also on the boundary of ϕ_n.

Conversely, there is *only* one such point, for let P'' be any

point on the boundary of ϕ_n equivalent to P, and let L' be the operation of Γ bringing P to coincide with P''.

Then, if P'' lies on the boundary of ϕ_n and ϕ_g, evidently L' brings ϕ_n to coincide with itself and ϕ_e to coincide with ϕ_g, or else it brings ϕ_n to coincide with ϕ_g and ϕ_e to coincide with ϕ_n. Now the first alternative is impossible unless $L'=1$, and the second is impossible unless $L'=L$, that is, unless P'' coincides with P'.

Finally, two equivalent points P and P' cannot be such that P lies on the surface of ϕ_u and P' inside ϕ_v; for if L is the operation of Γ bringing P to coincide with P', and if L brings ϕ_u to coincide with ϕ_w, then P' must lie on the surface of ϕ_w, and cannot therefore lie inside ϕ_v.

Hence we have proved that $\phi, \phi_1, \phi_2, \ldots$ are fundamental cells according to the definition of p. 241, if we consider a point lying on the surface between two of the series $\phi, \phi_1, \phi_2, \ldots$ as being half in one and half in the other.

§ 3. When the form and position of ϕ are chosen the forms and positions of the cells $\phi_1, \phi_2, \phi_3, \ldots$ are fixed, for they are obtained by transforming ϕ by the operations of Γ. They are all congruent to ϕ, or half congruent and half enantiomorphous to ϕ, according as Γ does not or does contain operations of the second sort. There is a considerable choice in obtaining the form and position of ϕ. This cell cannot, however, be pierced by a rotation-axis or symmetry-plane; for if that were possible a point inside the cell sufficiently close to the axis or plane would be brought by rotation about the axis or reflexion in the plane to coincide with a point also inside ϕ, so that ϕ would contain two equivalent points contrary to hypothesis. Rotation-axes and symmetry-planes must therefore lie in the surfaces of fundamental cells.

Examples of two divisions of space into fundamental cells for the group C^4_{2v} are given in Figs. 184 and 185. The point of view is that of Fig. 95 (p. 180); that is, the observer is supposed to be looking in the direction of $-\tau_z$. The position of the 2-al axes is shown by dots; the position of the other elements of symmetry may be found from Fig. 95.

In Fig. 184 the cell has a single boundary which may be taken as a cylinder whose height is $2\tau_z$ and whose base is any one of the areas there shown.

In Fig. 185 each cell has three boundaries and may be taken as made up of three cylinders whose height is $2\tau_z$, and whose bases are any three areas labelled with the same number.

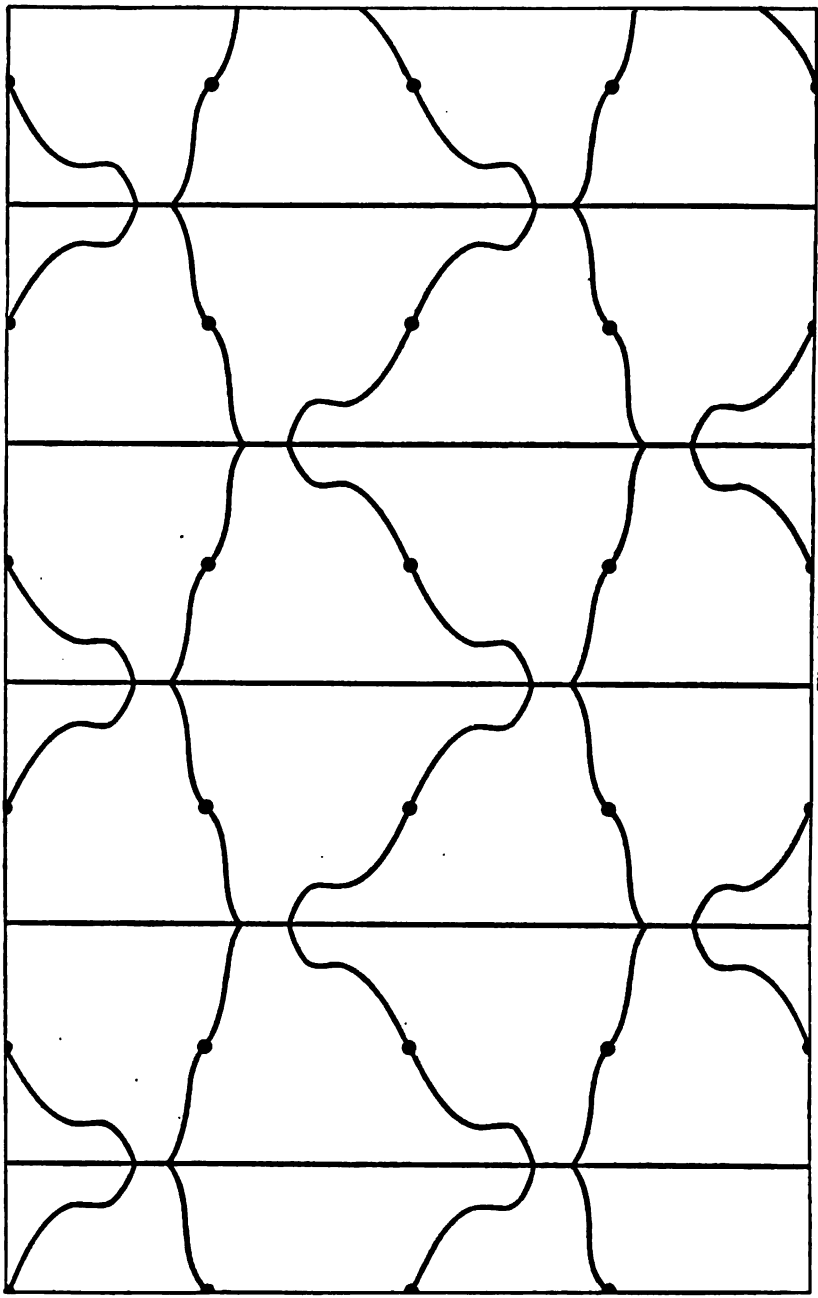

Fig. 184.

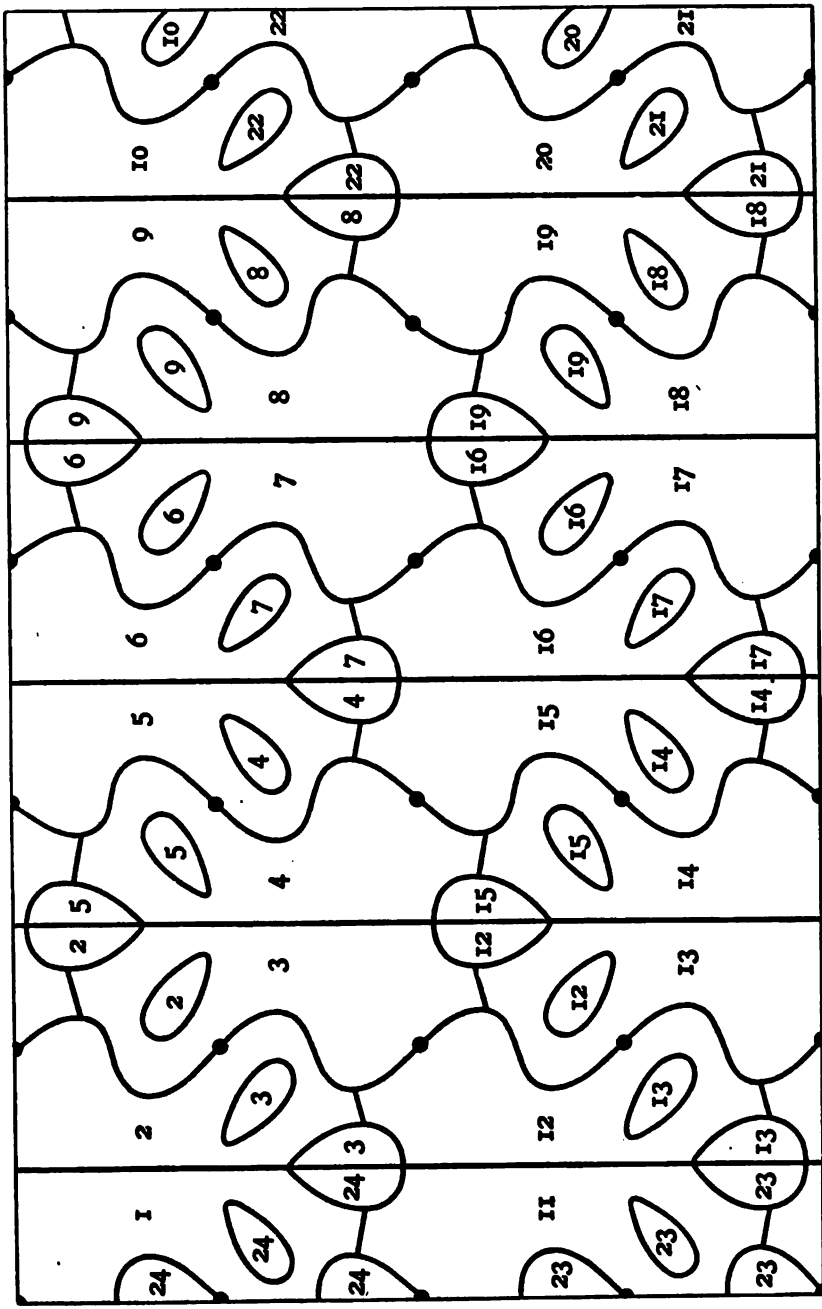

Fig 186.

The division of the cells into two enantiomorphous sets (due to the fact that C_{2v}^4 has operations of the second sort) is obvious in the figures; for example, in Fig. 185, cells 2, 3, 6, 7, &c., form a set congruent to one another and all enantiomorphous to 1, 4, 5, 8, &c.

It will be noticed that the rotation-axes lie in the boundaries, and that the symmetry-planes (but not necessarily the gliding-planes) form part of the boundaries of the fundamental cells.

§ 4. We have split up space into portions $\phi, \phi_1, \phi_2, \ldots$ such that each one is brought to coincidence with some other by every operation of Γ, and such that no one is brought to coincide with itself by any operation of Γ except unity. It is sometimes possible to divide up space into portions $\Phi, \Phi_1, \Phi_2, \ldots$ each of which is brought into coincidence with itself or with some other by every operation of Γ, and is brought to coincidence with itself by operations of Γ other than unity.

Let $1, L_1, L_2, L_3, \ldots, L_m$ be the operations of Γ which bring Φ to self-coincidence, then these operations must form a point-group for they all bring a *finite* figure to self-coincidence. Hence space cannot be divided into portions such as $\Phi, \Phi_1, \Phi_2, \ldots$ unless Γ has a point-group for a subgroup.

The converse is true that, if Γ has a point-group as a subgroup, regions such as $\Phi, \Phi_1, \Phi_2, \ldots$ can be found. For let the operations of a point-group, which is a subgroup of Γ, be $1, L_1, L_2, L_3, \ldots, L_m$ . . . . . (i), and let a fundamental cell ϕ be brought to coincide with $\phi, \phi_1, \phi_2, \ldots, \phi_m$ by these operations. Now take an operation M of Γ not included in (i), and let M bring these cells to coincide with the cells $\phi', \phi_1', \phi_2', \ldots \phi_m'$ respectively. Let the portion of space occupied by $\phi, \phi_1, \phi_2, \ldots, \phi_m$ taken together be denoted by Φ, and that occupied by $\phi', \phi_1', \phi_2', \ldots, \phi_m'$ by Φ_1. Now consider an operation R* of Γ which brings ϕ_e to coincide with ϕ_f; then $L_e.R$ brings ϕ to coincide with ϕ_f, and therefore $L_e.R = L_f$ or $R = L_e^{-1}.L_f$. Hence R brings ϕ_u to coincide with that cell to which ϕ is brought by $L_u.R$, that is, by $L_u.L_e^{-1}.L_f = L_p$, where L_p is one of the operations (i).

Therefore ϕ_u is brought by R to coincidence with ϕ_p, one of the cells composing Φ; and, since two different fundamental cells cannot be brought to coincide with the same

* It is easily seen that R belongs to the series (i).

cell by the same operation, Φ is brought to self-coincidence by every operation which brings one of the cells contained in it to coincidence with any other cell so contained.

Now consider an operation R_1 of Γ which brings ϕ_e to coincide with ϕ'_f; then, as before, $R_1 = L_e^{-1}.L_f.M$ (for $L_e^{-1}.L_f$ brings ϕ_e to coincide with ϕ_f and M brings ϕ_f to coincide with ϕ'_f). Hence R_1 brings ϕ_u to coincide with that cell with which ϕ is brought to coincide by $L_u.L_e^{-1}.L_f.M = L_p.M$, i.e. with a cell ϕ_p' contained in Φ_1. Hence Φ is brought to coincide with Φ_1 by every operation of Γ which brings one of the component cells of Φ to coincide with one of the component cells of Φ_1.

By taking other operations of Γ we can derive figures $\Phi_2, \Phi_3, \ldots$ and can fill all space with such figures. For if there is a fundamental cell ϕ_n not included in $\Phi, \Phi_1, \Phi_2, \Phi_3, \ldots$, we can always find an operation of Γ which brings ϕ to coincide with ϕ_n, derive a new figure from Φ by its aid, add this figure to $\Phi, \Phi_1, \Phi_2, \Phi_3, \ldots$, and repeat this process till all space is filled.

No two of these figures can overlap; for if Φ_q and Φ_r had a common cell ϕ_c, and if R were the operation of Γ bringing ϕ to coincide with ϕ_c, then R would bring Φ to coincide both with Φ_q and Φ_r which is impossible.

Lastly, any figure Φ_h is brought into coincidence with a similar figure by any operation E of Γ. For if ϕ is brought to coincide with any cell contained in Φ_h by the operation F of Γ, Φ is brought to coincide with Φ_h by F. But $F.E$ is an operation of Γ; hence Φ is brought by $F.E$, and therefore Φ_h by F, into coincidence with itself or some other figure of the series $\Phi, \Phi_1, \Phi_2, \ldots$.

This series therefore fills all space without overlapping, and is such that any one of the series is brought to coincidence with itself or some other figure of the series by any operation of Γ. Hence the theorem stated above is completely proved.

An interesting case is that in which Γ is formed by multiplying a translation-group by a point-group* whose operations are those of the row (i). $\Phi, \Phi_1, \Phi_2, \ldots$ are then derived by transforming any one of them by the translations. We have a partitioning of space into a series of similar and similarly orientated divisions. It may be shown that each such division is, in general, in contact with fourteen others; and may therefore be supposed to have fourteen walls (not

* That is, by combining the operations of a translation-group with those of a point-group.

in general plane). Each division must therefore 'be called a tetrakaidekahedron unless we prefer to call it a fourteen-walled cell *.'

If Γ has one point-group G as subgroup, it is easily seen that it must have an infinite number of point-groups G, G_1, G_2, G_3, ... as subgroups †. Any cell ϕ is in general differently situated with regard to the elements of each of these point-groups; hence a different region Φ is obtained from ϕ by the aid of each of G, G_1, G_2, G_3, ... Therefore Φ may have an infinite number of different forms corresponding to any given division of space into fundamental cells.

As an illustration we may take the division of space into fundamental cells for the group C_{3v}^2 which is shown in Fig. 186. The point of view is that of Fig. 162 (p. 221); the positions of the symmetry-elements are obvious. Each fundamental cell may be taken as a cylinder whose height is $2\tau_z$, and whose base is one of the numbered areas there shown.

In the following list are given a few of the infinite number of different ways of grouping the fundamental cells so as to form portions of space such as Φ, Φ_1, Φ_2, ..., corresponding to the various point-groups which are subgroups of C_{3v}^2; those cells which are enclosed in the same brackets belong to the same portion.

Point-group.	Groups of Fundamental Cells.
C_s	(1, 2) (3, 4) (5, 6) (7, 8) (9, 10) .....
	or (1, 4) (2, 5) (3, 6) (7, 10) (8, 11) .....
	or (6, 1) (2, 3) (4, 5) (12, 7) (8, 9) .....
	&c., &c.
C_3	(1, 3, 5) (2, 4, 6) (7, 9, 11) (8, 10, 12) .....
	or (5, 27, 31) (4, 12, 32) (11, 33, 37) (10, 18, 38) .....
	&c., &c.
C_{3v}	(1, 2, 3, 4, 5, 6) (7, 8, 9, 10, 11, 12) (13, 14, 15, 16, 17, 18) .....
	or (4, 5, 56, 57, 42, 37) (10, 11, 62, 63, 48, 43) (16, 17, 68, 69, 54, 49) ...
	&c., &c.

If the point-group is C_{3v} we have the particular case referred to at the bottom of p. 247.

* Lord Kelvin, Proc. Roy. Soc., lv (1894), p. 3.

† For let A be the system of symmetry-elements of G, and let A be brought to coincide with A' by any translation of Γ; then A' is a system of symmetry-elements of some point-group contained in Γ.

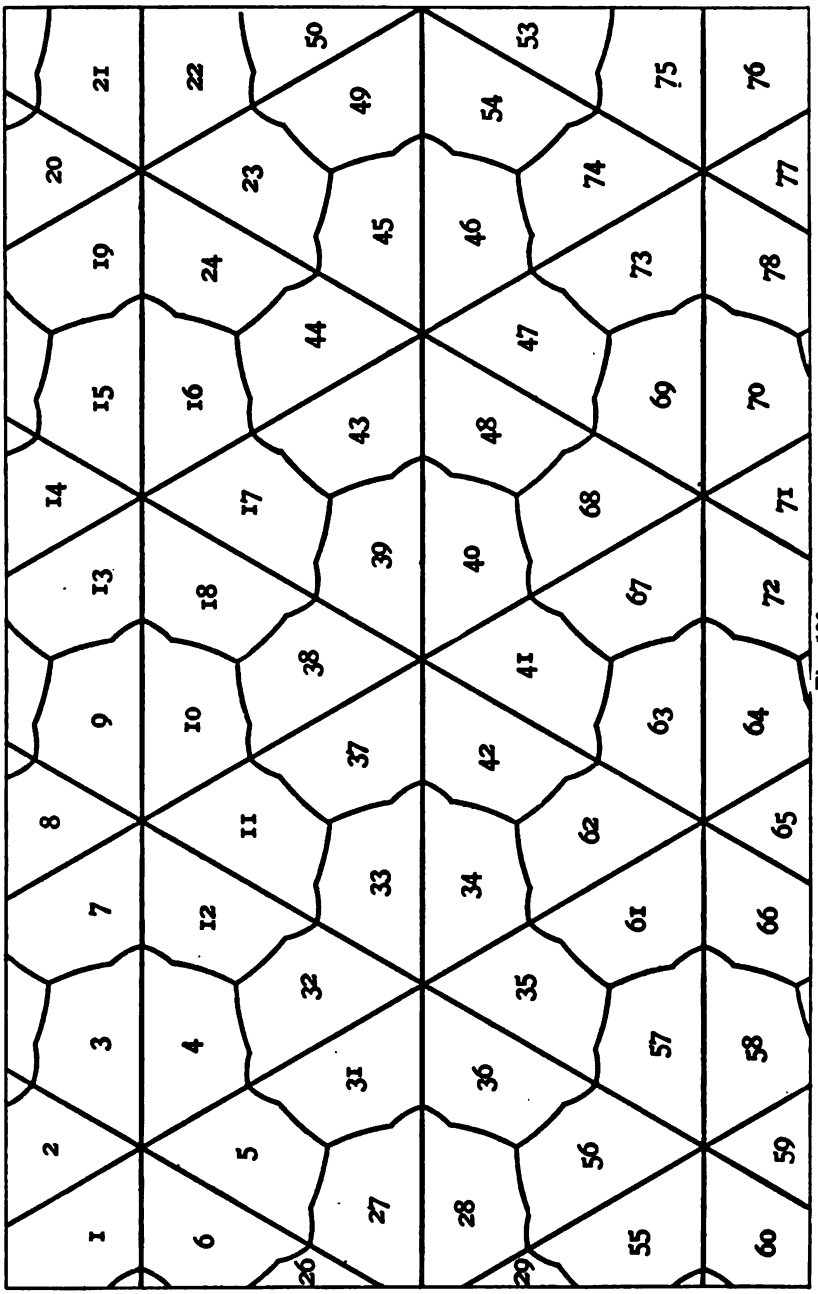
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Fig. 186.

CHAPTER XXV

CRYSTAL-MOLECULES.

§ 1. We have shown in chapters xviii to xxiii what are the different kinds of symmetry which an infinite collection of points capable of representing the structure of a crystalline medium may have on the hypothesis of p. 113.

This hypothesis led us easily to the statement* that such a collection is brought to self-coincidence by three independent finite translations, but by no infinitesimal translation. The most general collection of points capable of representing the structure of a crystalline medium is hence obtained by taking an arbitrary number of points, with arbitrarily chosen positions, and operating on them with all the movements (infinite in number) of one of the possible 230 space-groups. We thus obtain a collection of points whose symmetry is that of the group; and any arrangement of points which may be assumed as representing the structure of any particular crystal (provided such an arrangement has three independent finite translations and no infinitesimal translation) *must* be such a collection or a particular case of such a collection.

§ 2. Suppose that we have a crystalline medium whose physical symmetry is that of a point-group G. We shall assume that the system of molecules forming the medium has the symmetry of some space-group Γ isomorphous with G.

Our conception of the system of molecules is as follows: let any one molecule be represented by a cluster of points $P, Q, R, \ldots$ lying in a fundamental cell of Γ, then the points equivalent to $P, Q, R, \ldots$ which lie in any other fundamental cell are also supposed to represent a molecule. We have then a collection of molecules (one in each cell) which is brought to self-coincidence by every operation of Γ. If we suppose each fundamental cell to have only one boundary and the points $P, Q, R, \ldots$ to be close together in the cell, we can

* For proofs of this statement starting with a rather different hypothesis, see A. Schoenflies' "Krystallsysteme und Krystallstructur," part II, chap. xiv; K. Rohn, "Math. Annalen," liii (1900), p. 440; "Berichte der k. sächs. Gesellsch. der Wissensch. zu Leipzig," li (1899), p. 445.

form a clear mental picture of this. We may suppose, if we like (though this is quite unnecessary), that the points $P, Q, R, \ldots$ are the centres of a number of chemical atoms building up a chemical molecule, one of which is contained in each cell *.

Two such arrangements of points, four in each cell, represented by small circles of different sizes and shading † (those of the same size and shading being equivalent to one another), are shown in Figs. 187 and 188; the symmetry is that of C_{2v}^{8} and C_{4v}^{2} respectively.

The observer is supposed to be looking in the direction $-\tau_z$; the positions of the 2-al rotation-axes in Fig. 187 and of the 4-al rotation-axes in Fig. 188 are shown by crosses; the positions of the other elements of symmetry may be found from Figs. 99 and 137 (pp. 180, 207) respectively. All the circles of the same size and shading are supposed to represent points equally distant from the plane of the paper; the whole system of equivalent points is obtained by transforming those shown in the figure by the translations $2m\tau_z$ (m a positive or negative integer).

The individual molecules have no symmetry in general: they are all congruent if Γ has no operation of the second sort; if, however, Γ has such an operation the molecules may be divided into two sets, such that the molecules of either set are congruent to all the molecules of the same set and enantiomorphous to those of the other.

Some authors have objected to this division of the molecules of a crystal into two enantiomorphous sets as improbable. It is in fact in most cases possible to avoid having recourse to this division as follows:—we take (if possible) the points $P, Q, R, \ldots$ close together in a fundamental cell ϕ in such positions that the equivalent points $P_1, Q_1, R_1, \ldots$ in some cell enantiomorphous to and adjoining ϕ are close to $P, Q, R, \ldots$. We then consider $P, Q, R, \ldots, P_1, Q_1, R_1, \ldots$ as together representing a molecule.

Let $P, Q, R, \ldots$ be brought to coincidence with $P_a, Q_a, R_a, \ldots$,

* It must be remembered that by a 'molecule' is here meant 'that portion of matter which forms a unit in building up a crystalline (or amorphous) medium.' Thus by a 'molecule of quartz' we mean the smallest portion of matter which has a right to the name 'quartz.' It does not follow that no fraction of a 'quartz molecule' has a right to the name 'silica' (SiO_2), though it may be so.

† The points are at the centres of these circles. We may, if we prefer it, suppose the molecule represented by four spheres whose centres are these four points. Four is the smallest number of points (or spheres) which will show the difference between two enantiomorphous molecules.

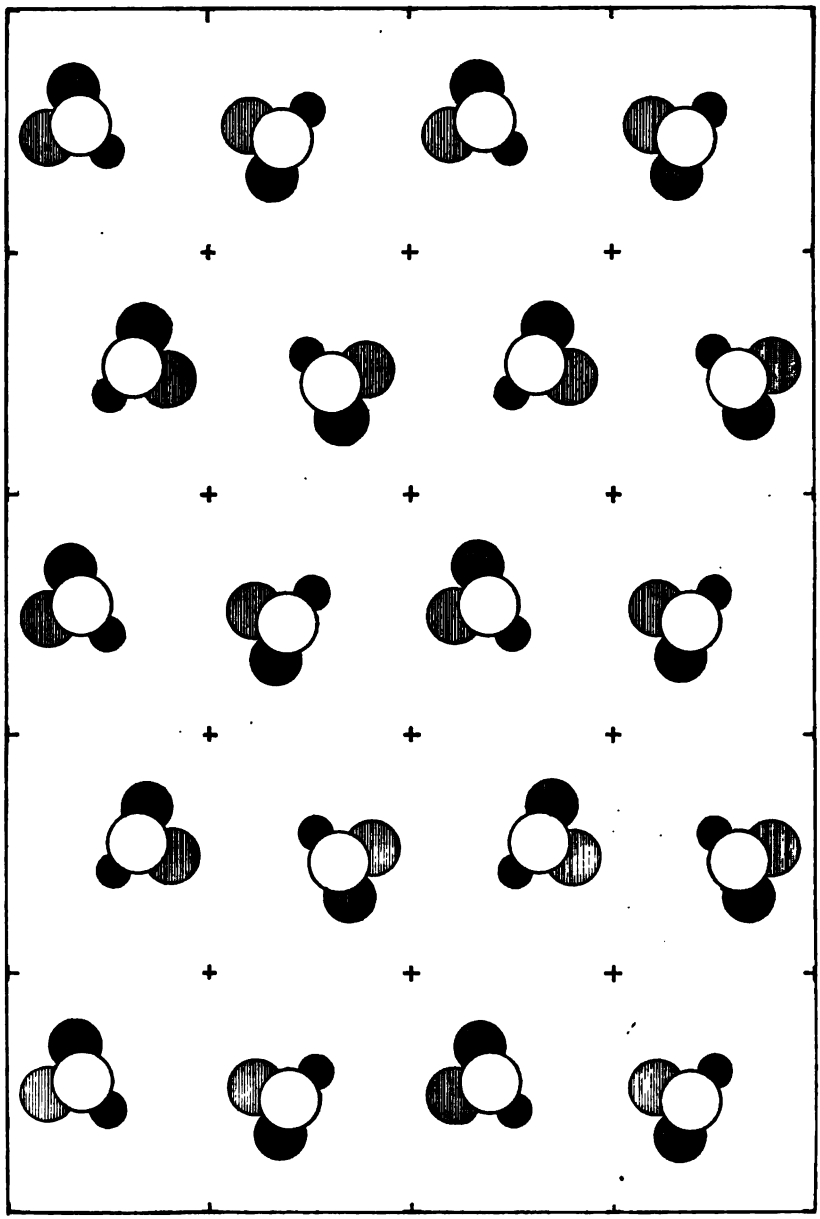

Fig. 187.

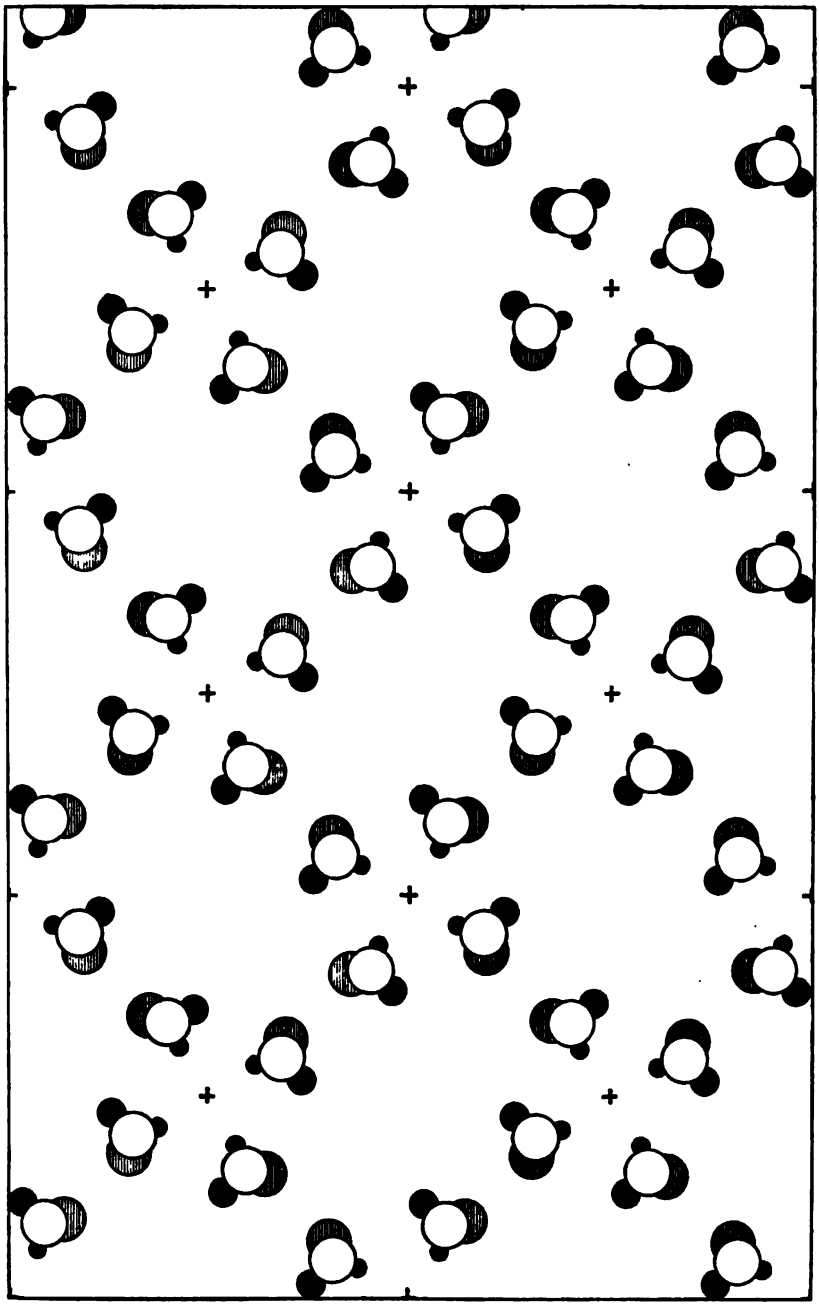

Fig. 188.

and $P_1, Q_1, R_1, \ldots$ with $P_b, Q_b, R_b, \ldots$ by an operation of the first sort contained in Γ; then $P_a, Q_a, R_a, \ldots, P_b, Q_b, R_b, \ldots$ together represent a molecule congruent to $P, Q, R, \ldots$, $P_1, Q_1, R_1, \ldots$, and proceeding in this way we can group all the points equivalent to $P, Q, R, \ldots$ into a series of *congruent* 'molecules.' The weakness of this method lies in the fact that it cannot be used to explain the structure of crystals of *every* class, without supposing that these congruent molecules may interlace with one another, an idea which hardly commends itself to a physicist; the method fails, for instance, in the case of C_{3i}^m.

§ 3. Now suppose that the group Γ can be derived by multiplying a translation-group by a *point-group* whose operations leave a point A unmoved. Let ϕ be a fundamental cell whose boundary passes through A, and suppose that ϕ is brought to coincide with $\phi, \phi_1, \phi_2, \ldots, \phi_m$ by the operations of the point-group. All other fundamental cells are obtained by transforming these by the translations.

Consider points $P, Q, R, \ldots$ in ϕ close to A, and let the points equivalent to them in $\phi_1, \phi_2, \ldots, \phi_m$ be

$$P_1, Q_1, R_1, \ldots; \; P_2, Q_2, R_2, \ldots; \; \ldots; \; P_m, Q_m, R_m, \ldots;$$

all these points are evidently close to A.

Then

$$P, Q, R, \ldots; \; P_1, Q_1, R_1, \ldots; \; P_2, Q_2, R_2, \ldots; \; \ldots; \; P_m, Q_m, R_m, \ldots;$$

may be considered as together forming a small 'molecule' μ close to A; this molecule has obviously the symmetry of the point-group.

Evidently in the same way all other points equivalent to $P, Q, R, \ldots$, may be grouped into 'molecules' derived from μ by transforming by the translations of the group.

This is Bravais' conception of a series of molecules each with the symmetry of a point-group, whose centres are arranged at the points of a lattice representing a translation-group. His theory has now been cleared from the accusation that it only defers the difficulty of explaining the symmetry of a crystal, and that it gives no adequate explanation of the assumed symmetry of the individual molecules (p. 143).

It has in fact been shown that, if his 'molecules' may be considered composite *, his structures are included among the most general arrangements of particles possible. His method has, however, the disadvantage that it is somewhat arbitrary;

* Bravais himself makes this assumption; see p. 143.

for he only makes use of a limited number of groups of movements, namely, the seventy-three which are obtained by multiplying a translation-group by a point-group; such as Q^1, Q^6, Q^7, Q^8 *.

In this and in the previous section we have, for the sake of clearness, supposed our points P, Q, R, ... finite in number and close together in a fundamental cell; this is not necessary, and we may take *any* points in a fundamental cell, and all points equivalent to them as representing the structure of a crystalline medium. The extreme cases are, on the one hand the case of a single point in each cell, on the other the case of an infinite number of points in each cell completely filling the cell. If we consider points infinite in number and completely filling a fundamental cell as together constituting a physical unit in making up the structure, we must take the cell with a single boundary; for we can hardly consider *physical* units as interlacing.

§ 4. It will be readily understood that our conception of crystal-structure is in accordance with the observed properties of matter, when we remember that our methods of observation do not enable us to determine the physical properties of a substance along a mathematical line, but only to determine the sum total of these properties along all the lines in a given direction which meet an area whose dimensions are large compared with the distances between neighbouring molecules. We should therefore expect, assuming the above theory of crystal-structure, that the physical properties in a fixed direction of all parts of a homogeneous crystalline medium would appear to be the same in spite of the fact that the material is coarse-grained. These considerations apply to an amorphous medium; but, while the geometrical symmetry of a crystalline medium would lead us to expect that its physical properties are different in different directions, there is no reason why we should anticipate this difference in the case of a medium which lacks this geometrical symmetry; we should therefore expect the physical properties of a homogeneous amorphous solid to be the same in every direction.

Since all the operations of any space-group are obtained by multiplying a finite number † of them by the translations (p. 158), therefore all the points of an equivalent system are

* Obtained by multiplying Q by Γ_o, Γ_o', Γ_o'', Γ_o''' respectively.

† This number is equal to the number of operations m in the isomorphous point-group.

obtained by transforming a finite number of them by the translations. This applies to the system equivalent to each of the points $P, Q, R, \ldots$ of § 2. Hence the series of points representing the constituent parts of a crystalline solid are the points of a series of similar and similarly orientated lattices * (any one of which represents the translations of the group).

We may therefore apply without appreciable modification Bravais' explanation of the cleavage, the occurrence of crystal faces, the law of rational indices, &c., as given in chapter XV.

§ 5. It must be remembered that all we have done is to obtain every structure which is *geometrically* possible, on the assumption that the structure has three independent finite translations, and no infinitesimal translation. It does not follow that every such structure is *mechanically* possible; that if particles were arranged in the way suggested by the geometrical investigation they would be in stable equilibrium. The geometrical theory is now complete, the mechanical theory is still in its infancy. That some particular cases of the general geometrical structure (if not the most general structure itself) are mechanically possible may be safely prophesied; further than this it is not at present possible to go †.

If the general symmetry of a crystal is that of a point-group G, its intimate structure has the symmetry of some space-group isomorphous with G; but it cannot yet be determined which of these isomorphous space-groups represents the intimate symmetry of *any* given material.

Attempts have been made to solve this problem in a few special cases; thus Viola assigns D^6_{3d} to Calcite and C^3_{2h} to Aragonite ‡. Fedorow allots O^2_h, T^4_d, T^2_h at high temperatures, and D^4_{4h}, D^1_{2d}, Q^2_h at ordinary temperatures to Leucite, Boracite, and Perowskite respectively. He bases this allotment on the optical behaviour of the substances, but further evidence is needed.

If we could find a crystal whose symmetry is that of C_{3h}, we could allot the space-group C^1_{3h} to it at once, since there is no other space-group isomorphous with C_{3h}; such a crystal, however, has not yet been found.

Crystals belonging to an enantiomorphous hemihedry can crystallize in two enantiomorphous forms (p. 92). In some cases two plates of similar direction and thickness cut from two

* Their number is mn, where n is the number of the points $P, Q, R, \ldots$.

† See Lord Kelvin's paper in the "Phil. Mag.," 1902, p. 149.

‡ "Zeitschr. f. Kryst. u. Min.," xxviii, p. 230.

such enantiomorphous crystals rotate the plane of polarized light through equal and opposite angles. It is natural to allot to two such enantiomorphous forms space-groups which are similar, except that if one has left-handed screws the other has right-handed, and vice versa *. For example, it is natural to assign to the two varieties of quartz either the two groups D_3^3 and D_3^5, or else the two groups D_3^4 and D_3^6; Sohncke has given reasons for adopting the latter alternative.

To crystals which rotate the plane of polarization and which belong to the regular enantiomorphous hemihedry we should allot the groups O_6 and O_7; no crystals, however, have yet been found belonging to this class which rotate the plane of polarization.

Among recent attempts to obtain the exact structure of certain crystals (not merely to assign to them the appropriate space-group) may be mentioned those of Barlow † and Sollas ‡. Both of these authors represent all atoms of the same element in a given crystal by spheres of the same size. The former chooses arrangements (consistent with the known physical symmetry of the crystals) in which the spheres are packed together as closely as possible. Professor Sollas, on the other hand, is guided by the observed density (and symmetry) in his selection of suitable arrangements. In the absence of any sufficiently developed mechanical theory which might demand closest packing, the latter method is perhaps to be preferred. Of course any one of the collections of spheres described by either author is brought to self-coincidence by every operation of one of the 230 possible space-groups.

* This is only possible with optically isotropic and uniaxial crystals. The suggestion cannot be applied to biaxial crystals, which rotate the plane of polarization, such as cane-sugar and rochelle salt ("Phil. Mag.," 1901, ii, p. 361); and the rotation in such cases must be ascribed to the constitution of the individual molecules.

† "Scientific Proceedings of the Royal Dublin Society," vol. viii, part vi, No. 62 (1897).

‡ "Proc. Roy. Soc.," lxiii, lxix.

CHAPTER XXVI

HISTORY OF THE STRUCTURE-THEORIES.

§ 1. A clear and full account of the earlier attempts to suggest an explanation of the internal structure of crystals is given in L. Sohncke's "Entwickelung einer Theorie der Krystallstruktur." Bravais' theory, which was a great step in advance, was put forward in 1850*. In 1869 C. Jordan published a note 'Sur les groupes de mouvements'†, in which he gives a full and complete proof of the existence of those groups which contain only operations of the first sort. He gives groups containing infinitesimal translations, and others involving only one or two independent translations, and it is by no means certain (though it is possible‡) that he realized the crystallographic application of his work. L. Sohncke, in 1879§, first clearly pointed out this application; his book deals only with the sixty-five groups which contain no operations of the second sort, and which have three independent and no infinitesimal translations.

At first he represented the structure of crystalline media by a system of points equivalent to a single point P. By considering structures, in which P has some special position with regard to the axes of symmetry, he was able to account for crystals whose physical symmetry is that of point-groups containing operations of the second sort. Later on he found this device unsatisfactory, and for the single point P he substituted n points‖; thus arriving practically at that method of representing the structure of a crystal by a system of *congruent* molecules which has been described on p. 251. To this method he adhered, for he considered Schoenflies' division of the molecules of a substance into two enantiomorphous sets extremely improbable.

On the *probability* of either method it is impossible to form an opinion in the absence of any mechanical theory; and

* Mémoire sur les systèmes formés par des points, &c., "Journ. de l'École Polyt.," xix, 33, p. 1.

† "Ann. di matematica pura ed applicata," ii, 2.

‡ His use of the terms 'merohedry,' &c., may be a sign that he had this application in view.

§ "Entwickelung einer Theorie der Krystallstruktur."

‖ Erweiterung der Theorie der Krystallstructur, "Zeitschr. f. Krystall. u. Min.," xiv, p. 485.

Schoenflies' contention that it is sufficient if his procedure is geometrically possible seems reasonable. Moreover, Sohncke's arguments were greatly weakened by the fact that he could not apply his method in every case (see p. 254).

The proof of the existence of all the 230 groups of movements is due to Fedorow, Schoenflies, and Barlow. Fedorow's and Schoenflies' work appeared about 1890; the former had a slight priority, but his work was published in Russian, and was, therefore, not so widely circulated as it might otherwise have been. He divided the 230 groups into 73 which are obtained by multiplying a translation-group by a point-group, 54 whose operations of the first sort only can be so derived, and 103 whose operations of the first sort cannot be so derived *. Q_h^1, Q_h^2, and Q_h^6 are examples of these three classes respectively.

Schoenflies published preliminary notes of his work in the "Math. Annalen" †, and in 1891 produced his "Krystallsysteme und Krystallstructur," in which he gives a full description of the groups, and the method of dividing space into 'fundamental cells.' We have in this book chosen lines of proof which are, in most cases, practically identical with those of Schoenflies.

Barlow published his work later ‡ but independently. He gives first those 65 groups which have only operations of the first sort (Sohncke's groups); then the 92 groups derived from them by multiplying by an inversion; then those 71 groups of the second sort which cannot be derived with the aid of an inversion, but which can be derived by multiplying by an operation isomorphous with a reflexion; and, lastly, those two groups which can only be derived by multiplying by a rotatory-reflexion ($C_4'^1$ and $C_4'^2$). He confines himself to a statement of the results arrived at without giving a full account of the methods employed §.

The geometrical theory of crystal-structure seems to be now fairly complete; it is probable that further advance is to be expected on the physical or mechanical side.

For a fuller historical account the reader is referred to the above quoted work of Sohncke, and to the British Association report on "The Structure of Crystals," 1901.

* He calls these groups symmorphous, hemi-symmorphous, and asymmorphous respectively.

† "Math. Ann.," xxviii, xxix, xxxiv.

‡ "Zeitschr. f. Kryst. u. Min.," xxiii, 1894.

§ For a comparison of the results of Fedorow, Schoenflies, and Barlow see "Phil. Mag.," Feb. 1902.

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THE END

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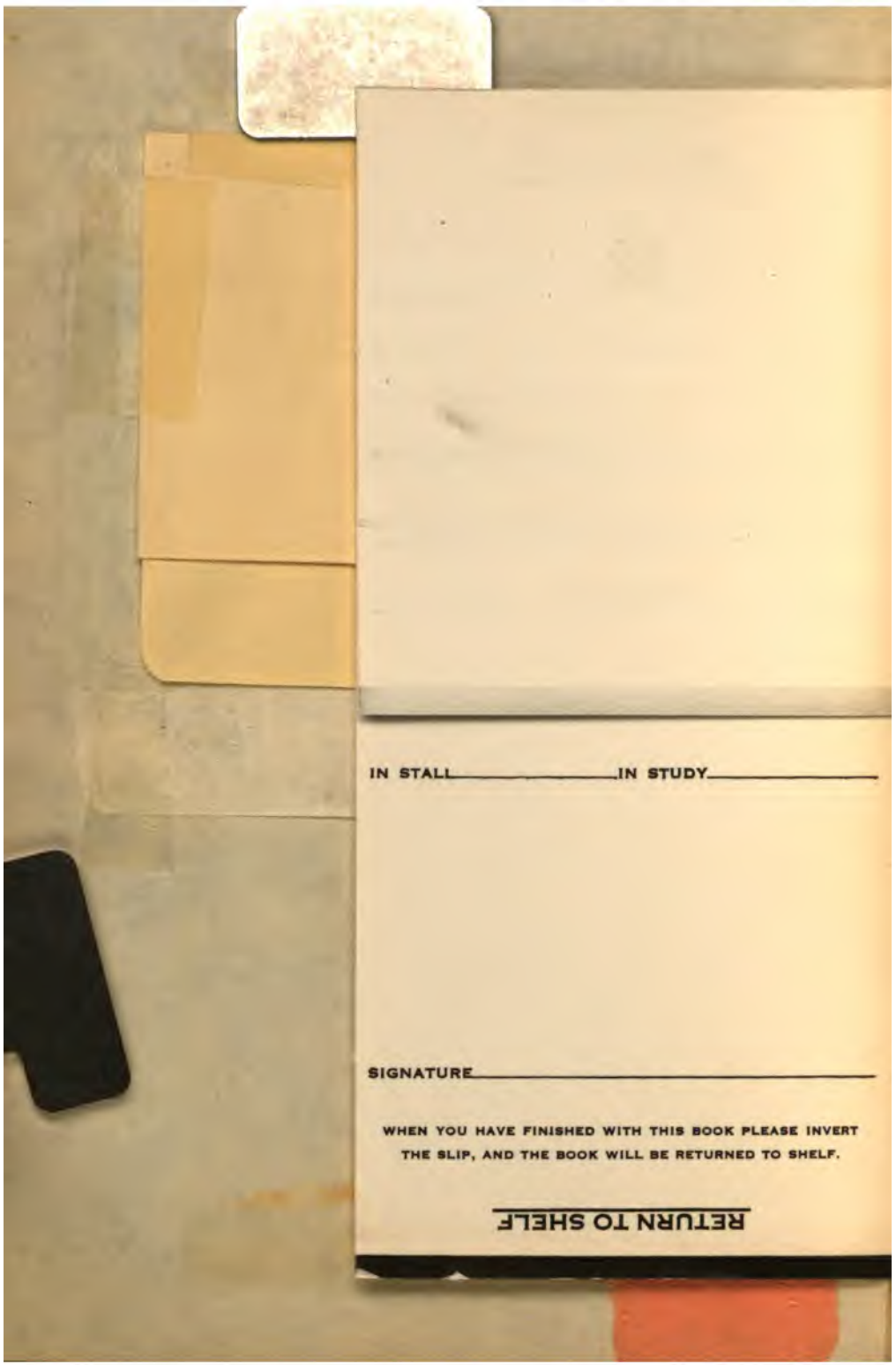
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